Sample Questions: Limits

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1. Let X_n have an exponential distribution with parameter $\lambda = n$. Pick one of these and prove your answer.
   - $X_n \xrightarrow{p} 0$
   - $X_n \xrightarrow{p} 1$
   - X_n does not converge in probability to a constant.

$E(X_n) = \frac{1}{\lambda} = \frac{1}{n} \to 0$

$F_{X_n}(x) = 1 - e^{-nx}$ for $x > 0$

$$\lim_{n\to\infty} P\{|X_n - 0| \geq \varepsilon\} = \lim_{n\to\infty} P\{|X_n| \geq \varepsilon\}$$

$$= \lim_{n\to\infty} P\{X_n \geq \varepsilon\} = \lim_{n\to\infty} \left(1 - F_{X_n}(\varepsilon)\right)$$

$$= \lim_{n\to\infty} e^{-n\varepsilon} = \lim_{n\to\infty} \frac{1}{e^{n\varepsilon}} = 0 \checkmark$$

2. Let X_n have an exponential distribution with parameter $\lambda = n$, and let $Y_n = \frac{5X_n+2}{X_n+1}$. To what target does Y_n converge in probability?

$X_n \xrightarrow{P} 0$ by Q1 & $g(x) = \frac{5x+2}{x+1}$

is continuous at $x=0$. So by continuous mapping

$$Y_n = g(X_n) \xrightarrow{P} g(0)$$

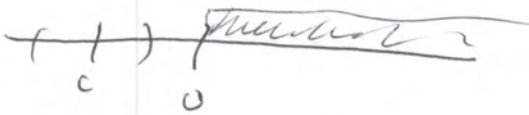
$$= \frac{5 \cdot 0 + 2}{0+1} = 2$$

So

$$Y_n \xrightarrow{P} 2$$

3. Let $X_n \sim$ Uniform$(0,n)$. Does X_n converge in probability to a constant? NO Because

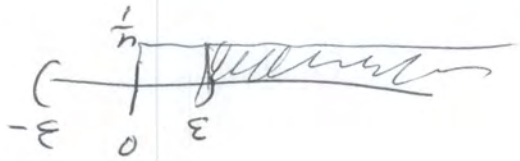
<u>If $c<0$</u>, $P\{|X_n - c| \geq \varepsilon\}$



Choose $\varepsilon = \frac{-c}{2}$ AND

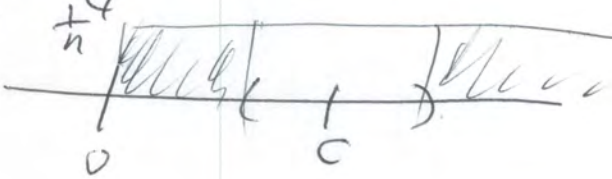
$$P\{|X_n - c| \geq \varepsilon\} = 1 \text{ NOT} \to 0$$

<u>If $c=0$</u> $P\{|X_n - c| \geq \varepsilon\}$



$$= 1 - \frac{\varepsilon}{n} \to 1 \text{ NOT zero}$$

If $c>0$, choose $\varepsilon < c$



$$\lim_{n\to\infty} P\{|X_n - c| \geq \varepsilon\}$$

$$= \lim_{n\to\infty}\left(1 - \frac{2\varepsilon}{n}\right) = 1 \neq 0$$

For $x \geq 0$ $F_{X_n}(x) = \frac{x}{n}$

4. Let $X_n \sim$ Uniform(0,n) and let $Y_n = \frac{X_n}{X_n+1}$. Prove $Y_n \xrightarrow{p} 1$.

$$\lim_{n\to\infty} P\left\{\left|\frac{X_n}{X_n+1} - 1\right| \geq \varepsilon\right\}$$

$$= \lim_{n\to\infty} P\left\{\left|\frac{X_n}{X_n+1} - \frac{X_n+1}{X_n+1}\right| \geq \varepsilon\right\}$$

$$= \lim_{n\to\infty} P\left\{\left|\frac{1}{X_n+1}\right| \geq \varepsilon\right\}$$

$$= \lim_{n\to\infty} P\left\{\frac{1}{X_n+1} \geq \varepsilon\right\} = \lim_{n\to\infty} P\left\{1 \geq X_n\varepsilon + \varepsilon\right.$$

$$= \lim_{n\to\infty} P\left\{X_n\varepsilon \leq 1-\varepsilon\right\} = \lim_{n\to\infty} P\left\{X_n \leq \frac{1-\varepsilon}{\varepsilon}\right\}$$

$$= \lim_{n\to\infty} F_{X_n}\left(\frac{1-\varepsilon}{\varepsilon}\right) = \lim_{n\to\infty}\left(\frac{1-\varepsilon}{n\varepsilon}\right) = 0$$

QED Quod Erat Demonstrandum

5. Let X be a random variable with expected value μ and variance σ^2, and let $Y_n = \frac{X}{n}$.

(a) Show $Y_n \xrightarrow{p} 0$. $E(Y_n) = E\left(\frac{X}{n}\right) = \frac{1}{n}E(X) = \frac{\mu}{n} \to 0$

$$\lim_{n\to\infty} Var(Y_n) = \lim_{n\to\infty} Var\left(\frac{X}{n}\right) = \lim_{n\to\infty} \frac{1}{n^2} Var(X)$$

$$= \lim_{n\to\infty} \frac{\sigma^2}{n^2} = 0$$

So $Y_n \xrightarrow{p} 0$ By Variance Rule.

(b) What if X is Cauchy? Does the result still hold? Okay sish

$$\lim_{n\to\infty} P\left\{\left|\frac{X}{n} - 0\right| \geq \varepsilon\right\} = \lim_{n\to\infty} P\left\{\left|\frac{X}{n}\right| \geq \varepsilon\right\}$$

$$= \lim_{n\to\infty} P\{|X| \geq n\varepsilon\} = \lim_{n\to\infty} \left(1 - P\{|X| < n\varepsilon\}\right)$$

$$= 1 - \lim_{n\to\infty} P\{-n\varepsilon < X < n\varepsilon\}$$

$$= 1 - \lim_{n\to\infty} \left(F_X(n\varepsilon) - F_X(-n\varepsilon)\right)$$

$$= 1 - \left(\lim_{n\to\infty} F_X(n\varepsilon) - \lim_{n\to\infty} F_X(-n\varepsilon)\right)$$

$$= 1 - (1 - 0) = 0 \quad \text{QED}$$

6. Let X_n have a Poisson distribution with parameter $n\lambda$, where $\lambda > 0$. This means $E(X_n) = Var(X_n) = n\lambda$. Let $Y_n = \frac{X_n}{n}$.

(a) For what values of y is $P(Y_n = y) > 0$?

$\{0, \frac{1}{n}, \frac{2}{n}, \ldots$

(b) Does Y_n converge in probability to a constant? Answer Yes or No and "prove" your answer.

$$E(Y_n) = E\left(\frac{X_n}{n}\right) = \frac{1}{n} E(X_n) = \frac{1}{\cancel{n}} \cancel{n}\lambda = \lambda$$

$$\to \lambda$$

$$Var(Y_n) = Var\left(\frac{X_n}{n}\right) =$$

$$= \frac{1}{n^2} Var(X_n) = \frac{1}{n^2} n\lambda = \frac{\lambda}{n} \to 0$$

So $X_n \xrightarrow{P} \lambda$ by Variance Rule

7. Let the discrete random variable X_n have probability mass function

$$p_{X_n}(x) = \begin{cases} \frac{1}{n} & \text{for } x = -n \\ \frac{n-2}{n} & \text{for } x = 0 \\ \frac{1}{n} & \text{for } x = n \end{cases}$$

(a) Try using the variance rule. $E(X_n) = 0$

$$Var(X_n) = E\{(X_n - E(X_n))^2\} = \sum_x (x-0)^2 p_X(x)$$

$$= (-n-0)^2 \cdot \frac{1}{n} + (0-0)^2 \frac{n-2}{n} + (n-0)^2 \cdot \frac{1}{n}$$

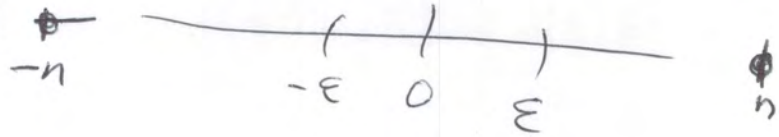
(b) Prove $X_n \xrightarrow{p} 0$ anyway.

$$= \frac{n^2}{n} + 0 + \frac{n^2}{n} = 2n$$

Let $0 < \varepsilon < 1$

$$\lim_{n\to\infty} P\{|X_n - 0| \geq \varepsilon\} = \lim_{n\to\infty} P\{|X_n| \geq \varepsilon\}$$

$$= \lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n}\right)$$



$$= \lim_{n\to\infty} \frac{2}{n} = 0$$

8. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables from an exponential distribution with parameter λ, so that $\boxed{E(X_i) = 1/\lambda}$ and $Var(X_i) = 1/\lambda^2$. Let $T_n = \frac{n}{\sum_{i=1}^{n} X_i}$. Show $T_n \xrightarrow{p} \lambda$.

By LLN $\bar{X}_n \xrightarrow{p} E(X_i) = \frac{1}{\lambda}$ & the function $g(x) = \frac{1}{x}$ is continuous at $x = \frac{1}{\lambda}$ and $T_n = 1/\bar{X}_n \xrightarrow{p} \frac{1}{E(X_i)} = \frac{1}{1/\lambda} = \lambda$

By continuous mapping. $\blacksquare$

9. Let the pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$ be selected independently from a joint distribution with $E(X_i) = \mu_x$, $E(Y_i) = \mu_y$, $Var(X_i) = \sigma_x^2$, $Var(Y_i) = \sigma_y^2$, and $Cov(X_i, Y_i) = \sigma_{xy}$. Independence means that X_i and Y_i are independent of X_j and Y_j for $i \neq j$.

Let $Z_i = aX_i + bY_i$. You would expect $\overline{Z}_n \xrightarrow{p} a\mu_x + b\mu_y$. Is this true? Answer Yes or No and say why.

Oh well, $E(Z_i) = aE(X_i) + b\,E(Y_i)$

$= a\mu_x + b\mu_y$, and

$Z_n \xrightarrow{p} a\mu_x + b\mu_y$ by LLN

This was not a very interesting question.

10. Let $f_{X_n}(x) = \begin{cases} \frac{n+1}{n} x^{1/n} & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

We have $X_n \xrightarrow{d} X$. Find the distribution of the target random variable X from the definition.

For $0 \le x \le 1$

$$F_{X_n}(x) = \int_0^x \frac{n+1}{n} t^{1/n}\, dt = \left(\frac{n+1}{n}\right) \frac{t^{\frac{1}{n}+1}}{\frac{1}{n}+1} \Bigg|_0^x$$

$$= \left(\cancel{\frac{n+1}{n}}\right) \frac{x^{\frac{1}{n}+\frac{n}{n}}}{\cancel{\frac{n+1}{n}}} = x^{\frac{n+1}{n}}$$

$$\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} x^{\frac{n+1}{n}} = x^{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)} = x$$

So

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

UNIFORM(0,1)

target

11. Let $p_{X_n}(x) = \begin{cases} \frac{n+3}{2(n+1)} & \text{for } x = 0 \\ \frac{n-1}{2(n+1)} & \text{for } x = 1 \end{cases}$

Show that $X_n \xrightarrow{d} X \sim \text{Bernoulli}(\theta \Leftarrow \frac{1}{2})$.

Check $F_{X_n}(x)$ at $x<0$, $0<x<1$, $x>1$

For $x<0$, $\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} 0 = 0$

For $0<x<1$, $\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} P(X_n=0)$

$$= \lim_{n\to\infty} \frac{n+3}{2(n+1)} \cdot \frac{1/n}{1/n}$$

$$= \lim_{n\to\infty} \frac{1+3/n}{2(1+1/n)} = \frac{1}{2}$$

For $x>1$, $\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \left(P(X_n=0) + P(X_n=1)\right)$

$$= \lim_{n\to\infty} \frac{n+3+n-1}{2(n+1)} = \lim_{n\to\infty} \frac{2n+2}{2(n+1)}$$

$$= \lim_{n\to\infty} 1 = 1$$

Agrees with Bernoulli $(\frac{1}{2})$ at all points of continuity

$$\Phi\left(\frac{x-\mu}{\sqrt{n}}\right)$$

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

12. Let $X_n \sim \text{Normal}(\mu, n)$. Does X_n converge in distribution to a random variable? Answer Yes or No and show your work.

$$F_{X_n}(x) = P(X_n \le x) = P\left(\frac{X_n-\mu}{\sigma} \le \frac{x-\mu}{\sqrt{n}}\right)$$

$$= F_Z\left(\frac{x-\mu}{\sqrt{n}}\right)$$

$$\lim_{n\to\infty} F_Z\left(\frac{x-\mu}{\sqrt{n}}\right) \overset{\text{cont.}}{=} F_Z\left(\lim_{n\to\infty} \frac{x-\mu}{\sqrt{n}}\right)$$

$$= F_Z(0)$$

$$= \frac{1}{2}$$

13. Let X be a degenerate random variable with $P(X = c) = 1$.

(a) Sketch $F_X(x)$.

(b) Find the moment-generating function of X.

$$E(e^{xt}) = \sum_{x;\, p(x)>0} e^{xt} p(x) = e^{ct} \cdot 1$$

$$= e^{ct}$$

14. Let X_n be an exponential random variable with parameter $\lambda = n$. It seems that $X_n \xrightarrow{d} X$. Find the distribution of the target random variable X

(a) From the definition. X is degenerate at zero

For $x < 0$, $\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} 0 = 0$

For $x > 0$ $\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} (1 - e^{-nx})$

$$= 1 - \lim_{n\to\infty} \frac{1}{e^{nx}} = 1 - 0 = 1$$

compare 13(a)

(b) Using moment-generating functions.

$$\lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^{-1} = \lim_{n\to\infty} \frac{1}{1 - \frac{t}{n}} = 1$$

$= e^{0t}$, MGF of RV degenerate at $x = 0$. Compare Q 13 b

15. Use moment-generating functions to prove the Law of Large Numbers.

Let $X_1, X_2, \ldots$ be independent random variables with common MGF $M(t)$. Then

$$M_{\bar{X}_n}(t) = M_{\sum_{i=1}^n X_i / n}(t) = M_{\sum_{i=1}^n X_i}\left(\frac{t}{n}\right)$$

$$\overset{\text{ind}}{=} \prod_{i=1}^n M_{X_i}\left(\frac{t}{n}\right) = M\left(\frac{t}{n}\right)^n, \text{ and}$$

$$\lim_{n\to\infty} M_{\bar{X}_n}(t) = \lim_{n\to\infty} M\left(\frac{t}{n}\right)^n$$

$$= \lim_{n\to\infty} \operatorname{Exp}\left\{\ln\left(M\left(\frac{t}{n}\right)^n\right)\right\}$$

$$= \operatorname{Exp}\left\{\lim_{n\to\infty} n \ln M\left(\frac{t}{n}\right)\right\}$$

$$= \operatorname{Exp}\left\{\lim_{n\to\infty} \frac{\ln M\left(\frac{t}{n}\right)}{n^{-1}}\right\}$$ Form $\frac{0}{0}$ use L'Hôpital's rule

$$= \operatorname{Exp}\left\{\lim_{x\to\infty} \frac{\frac{d}{dx} \ln M(tx^{-1})}{\frac{d}{dx} x^{-1}}\right\}$$

$$= \operatorname{Exp}\left\{\lim_{x\to\infty} \frac{M'\left(\frac{t}{x}\right) \cdot t \cdot \cancel{(-x^{-2})}}{M\left(\frac{t}{x}\right) \cancel{(-x^{-2})}}\right\}$$

$$= \operatorname{Exp}\left\{t \frac{\lim_{x\to\infty} M'\left(\frac{t}{x}\right)}{\lim_{x\to\infty} M\left(\frac{t}{x}\right)}\right\} =$$

15 continued

continuity

$$= \operatorname{Exp}\left\{ t \frac{M'\left(\lim_{x\to\infty} \frac{t}{x}\right)}{M\left(\lim_{x\to\infty} \frac{t}{x}\right)} \right\}$$

$$= \operatorname{Exp}\left\{ t \frac{M'(0)}{M(0)} \right\} = \operatorname{Exp}\left\{ \frac{t\mu}{1} \right\}$$

$= e^{\mu t}$, MGF of RV degenerate at μ,

So $\bar{X}_n \xrightarrow{d} \mu \Rightarrow \bar{X}_n \xrightarrow{p} \mu$,

which is the Law of Large Numbers

16. Let S be the sum of 16 independent Uniform(0,1) random variables. Find the approximate $P(S > 12)$. You may use the fact that a Uniform(0,1) has expected value $\frac{1}{2}$ and variance $\frac{1}{12}$.

$$P(S > 12) = P\left(\sum_{i=1}^{16} X_i > 12\right) = P\left(\bar{X}_n > \frac{12}{16}\right)$$

$$= P\left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} > \frac{\sqrt{16}\,(.75 - \frac{1}{2})}{\sqrt{1/12}}\right)$$

$$= P(Z_n > 0.\cancel{653}) \quad 3.46$$

$\approx \cancel{0.258}$

0.0003

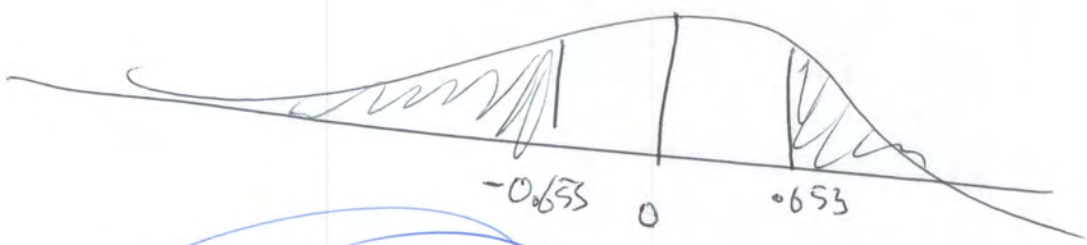
17. A multiple choice test has 50 questions with answers ABCD. If a student answers completely at random, what are the chances of getting 15 or more correct? You may use the fact that a Bernoulli(θ) has expected value θ and variance $\theta(1-\theta)$.

$X_i \sim \text{Bernoulli}\left(\theta = \frac{1}{4}\right)$

$$P\left(\sum_{i=1}^{50} X_i \geq 15\right) = P\left(\sum_{i=1}^{50} X_i > 14.5\right)$$

$$= P\left(\bar{X}_n > \frac{14.5}{50}\right) = P\left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} > \frac{\sqrt{50}(.29-.25)}{\sqrt{.25 * .75}}\right)$$

$$= P(Z_n > 0.653) \approx P(Z > 0.653)$$



$$= 0.258$$

max

18. In a walk-in medical clinic, the time a doctor spends per patient (including paperwork) comes from an unfamiliar skewed distribution with mean 5.1 and standard deviation 4.8 minutes. Find the maximum number of patients that ~~should~~ be scheduled so that the probability of working more than a 7 hour day will be less than 5%.

Forget it. This is too messy.

The answer is $n = 69$.

This handout was prepared by Jerry Brunner, Department of Mathematical and Computational Sciences, University of Toronto. It is licensed under a Creative Commons Attribution - ShareAlike 3.0 Unported License. Use any part of it as you like and share the result freely. The LaTeX source code is available from the course website:

http://www.utstat.toronto.edu/~brunner/oldclass/256f19