Def bond valuation using PIDE:

a) recovery of R at default time τ.

constant hazard rate (λ) and interest rate (r)

$$\bar{P}_t(T) = \mathbb{E}_t\left[e^{-r(T-t)} \mathbb{1}_{\tau > T} + R e^{-r(\tau-t)} \mathbb{1}_{\tau \le T} \right]$$

$$e^{-rt} \bar{P}_t(T) = \mathbb{E}_t\Big[\underbrace{e^{-rT} \mathbb{1}_{\tau > T} + R e^{-r\tau} \mathbb{1}_{\tau \le T}}_{\mathcal{F}_T\text{ - measurable}} \Big]$$

$\Rightarrow g_t \triangleq e^{-rt} \bar{P}_t(T)$ is a $\mathbb{Q}$-mtg. but depends on N_t – the Poisson jump.

write $g_t = g(t, N_t)$, then

$$dg = \partial_t g \, dt + [\underbrace{g(t, N_{t-}+1)}_{R e^{-rt}} - g(t, N_{t-})] \, dN_t$$

only interested in $N_t = 0$ at 1

$\Rightarrow E_t[dg_t] = 0 \Rightarrow$

$$\begin{cases} \partial_t g(t,0) + \lambda (R e^{-rt} - g(t,0)) = 0 \\ \text{and } g(T,0) = 1 \end{cases}$$

can solve easily ... show match expectation result.

b) recovery of $\alpha P_{\tau-}(T)$ at default τ – i.e. a percentage of pre-def. value.

$$\tilde{P}_t(T) = \mathbb{E}_t\left[e^{-r(T-t)} \mathbb{1}_{\tau > T} + \alpha P_{\tau-}(T) e^{-r(\tau-t)} \mathbb{1}_{\tau \le T} \right]$$

again $g_t = e^{-rt} \tilde{P}_t(T)$ is a $\mathbb{Q}$-mtg. $g_t = g(t, N_t)$

$$\Rightarrow \quad dg = \partial_t g \, dt + [\underbrace{g(t, N_{t-}+1)}_{\alpha\, g(t,0)\, e^{-rt}} - g(t, N_{t-})] \, dN_t$$

$$\mathbb{E}_t[dg_t] = 0$$

$$\Rightarrow \quad \partial_t g(t,0) + \lambda\left(\alpha\, g(t,0)\, e^{-rt} - g(t,0)\right) = 0 \quad , \quad g(T,0) = 1$$

$$\Rightarrow \quad \partial_t g(t,0) + g(t,0) \; \lambda\,(\alpha e^{-rt} - 1) = 0$$

$$\Rightarrow \quad \partial_t \ln g = \lambda(1 - \alpha e^{-rt})$$

$$g(T,0) - g(t,0) = \lambda(T-t) - \frac{\lambda\alpha}{r}\left(1 - e^{-r(T-t)}\right)$$

$$\Rightarrow \quad g(t,0) = \exp\left\{ -\lambda(T-t) + \frac{\lambda\alpha}{r}(1 - e^{-r(T-t)}) \right\}$$

$$\tilde{P}_t(T) = \mathbb{1}_{\tau > t} \cdot \exp\left\{ -(\lambda + r)(T-t) + \underbrace{\frac{\lambda\alpha}{r}(1 - e^{-r(T-t)})}_{(A)} \right\}$$

notice $(A) \xrightarrow[r \to 0]{} \lambda\alpha$

so $\tilde{P}_t(T) \longrightarrow \mathbb{1}_{\tau > t} \cdot e^{-\underline{\underline{\lambda(1-\alpha)}}(T-t)}$

hazard reduced by recovery %. α.

c) Poisson change of measure...

M_t — λ hazard rate under $\mathbb{P}$

suppose

$$\frac{d\eta_t}{\eta_{t-}} = a\left(dN_t - \lambda\, dt\right) \qquad \text{— clearly a } \mathbb{P}\text{-mtg.}$$

$$\eta_0 = 1$$

now easy to see that

$$\eta_t = e^{-a\lambda t}\,(1+a)^{N_t}$$

$$= \exp\{ -a\lambda t + \ln(1+a) \, N_t \}$$

η_t can induce a measure change! i) $\eta_0 = 1$

ii) η_t is a $\mathbb{P}$-mtg

iii) $\eta_t > 0$ a.s.

$$\left(\frac{d\mathbb{P}_a^*}{d\mathbb{P}}\right) \triangleq \eta_T \quad \text{on } (\Omega, \mathbb{P}, \mathbb{F}) \quad \mathbb{F} = \{(\mathcal{F}_t^N)_{0 \le t \le T}\}.$$

what is N_t in $\mathbb{P}_a^*$ measure?

know N_t must still be jump process of jump size 1

but what about activity?

$$\mathbb{E}^{\mathbb{P}_a^*}[e^{uN_T}] = \mathbb{E}^{\mathbb{P}}[e^{uN_T} \eta_T]$$

$$= \mathbb{E}^{\mathbb{P}}[e^{uN_T} \cdot e^{-\lambda a T + \ln(1+a) N_T}]$$

$$= e^{-\lambda a T} \, \mathbb{E}^{\mathbb{P}}[e^{(u + \ln(1+a)) N_T}]$$

$$= e^{-\lambda a T} \exp\{(e^{(u+\ln(1+a))} - 1)\lambda T\}$$

$$= \exp\{\lambda[-a + (e^u \cdot (1+a) - 1)] T\}$$

$$= \exp\{\lambda(e^u - 1)(1+a) T\}$$

$$= e^{\lambda_a^* (e^u - 1) T}, \quad \lambda_a^* = \lambda(1+a)$$

N_T is a $\mathbb{P}_a^*$ - Poisson process with hazard $\lambda_a^* = \lambda(1+a)$!

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$$\frac{d\eta_t}{\eta_{t-}} = a\,(dN_t - \lambda\,dt)$$

if jump

$$\eta_{t-} \longrightarrow \underbrace{\eta_{t-}(1+a)}_{\eta_t}$$

$$d\ln\eta_t = \underbrace{\frac{1}{\eta_{t-}}\,d\eta_t^c}_{-a\lambda\,dt} + (\Delta\ln\eta_t)\,dN_t$$

$$\Delta\ln\eta_t = \ln\eta_t - \ln\eta_{t-}$$

$$\ln(1+a)$$

$$d\eta_t = a\,\eta_{t-}\,dN_t$$

jump $\quad \eta_t - \eta_{t-} = a\,\eta_{t-} \Rightarrow \eta_t = \eta_{t-}(1+a)$

$$\ln\eta_t - \ln\eta_{t-} = \ln(\eta_{t-}(1+a)) - \ln\eta_{t-}$$

$$= \ln(1+a)$$

integrate both sides

$$\ln\eta_t - \ln\eta_0 = -a\lambda t + \ln(1+a)(N_t - \overset{0}{N_0})$$

$$\eta_t = \eta_0\,e^{-a\lambda t + \ln(1+a)N_t}$$

$$= \eta_0\,e^{-a\lambda t}\,(1+a)^{N_t}$$

$$d\left(e^{-\lambda a t + \ln(1+a)N_t}\right)$$

$$= d(e^{-\lambda a t})\,e^{\ln(1+a)N_t} + e^{-\lambda a t}\,d(e^{\ln(1+a)N_t})$$

$$= -\lambda a\,dt\,\eta_t$$

$$+ e^{-\lambda a t}\left(e^{\ln(1+a)(N_{t-}+1)} - e^{\ln(1+a)N_{t-}}\right)dN_t$$

$$\hookrightarrow \quad \underbrace{e^{-\lambda a t} \; e^{\ln(1+a) N_{t-}}}_{\eta_{t-}} \; \underbrace{\left(e^{\ln(1+a)} - 1 \right)}_{a} \; dN_t$$