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Short rate models

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\,dW_t \qquad \text{Vasicek}$$

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\sqrt{r_t}\,dW_t \qquad \text{CIR}$$

$$P_t(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,ds}\right] = e^{A_t(T) - B_t(T)\,r_t}$$

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[(P_T(u) - K)_+\, e^{-\int_t^T r_s\,ds}\right]$$

$$= P_t(T)\,\mathbb{E}^{\mathbb{Q}_T}\left[\frac{(P_T(u) - K)_+}{\underbrace{P_T(T)}_{\hookrightarrow 1}}\right]$$

$P_T(u)/P_T(T) \rightarrow X_t = \dfrac{P_t(u)}{P_t(T)}$

$$= P_t(T)\,\mathbb{E}^{\mathbb{Q}_T}\left[(X_T - K)_+\right]$$

$$\frac{dX_t}{X_t} = \sigma\left(-B_t(u) + B_t(T)\right)dW_t$$

X_T still log-normal distributed.

Add jumps ...

$$dr_t = \kappa(\theta - r_{t-})\,dt + \sigma\,dW_t + dJ_t$$

$$J_t = \sum_{n=1}^{N_t} j_n \qquad j_1, j_2, \ldots \text{ iid} \sim F(y)$$

N_t is Poisson w/ rate λ.

$$r_t = e^{-\kappa t}\,y_t$$

$$dr_t = -\kappa\,dt\,r_{t-} + e^{-\kappa t}\,dy_t$$

$$\Rightarrow e^{-\kappa t}\,dy_t = \kappa\theta\,dt + \sigma\,dW_t + dJ_t$$

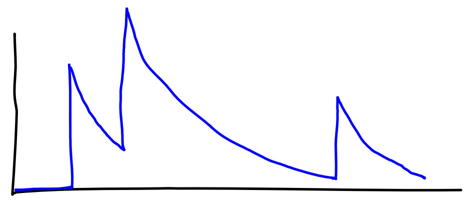
$$dy_t = \kappa\theta e^{\kappa t}\,dt + \sigma e^{\kappa t}\,dW_t + e^{\kappa t}\,dJ_t$$

$$y_t - y_0 = \theta(e^{\kappa t}-1) + \sigma\int_0^t e^{\kappa u}\,dW_u + \int_0^t e^{\kappa u}\,dJ_u$$

$\hookrightarrow r_0$

$$\Rightarrow r_t = r_0 e^{-\kappa t} + \theta(1-e^{-\kappa t}) + \sigma\int_0^t e^{-\kappa(t-u)}\,dW_u + \int_0^t e^{-\kappa(t-u)}\,dJ_u$$

$$\sum_{n=1}^{N_t} e^{-\kappa(t-t_n)}\, j_n$$



$$P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,ds}\right]$$

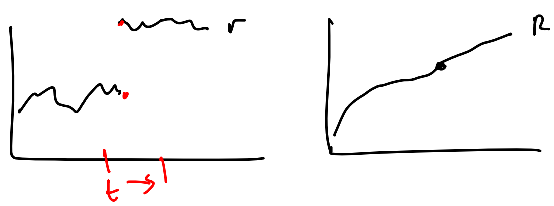
$$\Rightarrow e^{-\int_0^t r_s\,ds}P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_0^T r_s\,ds}\right] = h_t$$

$$h_t = h(t, r_t, \underbrace{\int_0^t r_s\,ds}_{R_t})$$

$dR_t = r_t\,dt$

$$dh_t = (\partial_t + \mathcal{L})h\,dt + \sigma\,\partial_r h\,dW_t + \left(h(t, r_{t-}+j_{N_t}, R_t) - h(t, r_{t-}, R_{t-})\right)dN_t$$

$$\mathcal{L} = \kappa(\theta - r)\partial_r + \tfrac{1}{2}\sigma^2\partial_{rr} + r\,\partial_R$$



$$\mathbb{E}_t[dh_t] = 0$$

$$\Rightarrow (\partial_t + \mathcal{L})h + \int_{-\infty}^{\infty}\left(h(t, r+y, R) - h(t, r, R)\right)dF(y)\cdot\lambda = 0$$

$$h = e^{A_t - B_t r - C_t R}$$

$$(\dot{A} - \dot{B} r - \dot{C} R)$$

$$+ \kappa(\theta - r)(-B) + \frac{1}{2}\sigma^2(-B)^2 + r(-C)$$

$$+ \int_{-\infty}^{\infty} \left(e^{-B_t y} - 1\right) dF(y)\, \lambda = 0 , \qquad h_T = e^{-R}$$

$$\Rightarrow A_T = 0, \quad B_T = 0, \quad C_T = 1$$

$$\Rightarrow \quad \dot{C} = 0 \quad \Rightarrow \quad C = \text{const.} = 1$$

$$-\dot{B} + \kappa B - \underbrace{C}_{1} = 0 \qquad B = \frac{1 - e^{-\kappa(T-t)}}{\kappa}$$

$$\dot{A} - \kappa\theta B + \frac{1}{2}\sigma^2 B^2 + \lambda \int_{-\infty}^{\infty} \left(e^{-B_t y} - 1\right) dF(y) = 0$$

$$\lambda \int_t^T \left(\int_{-\infty}^{\infty} \left(e^{-B_u y} - 1\right) dF(y) \right) du$$

$$h = e^{A_t - B_t r - R}$$

$$h = e^{-R} P \quad \Rightarrow \quad P = e^{A_t - B_t r}$$

How to value a bond option?

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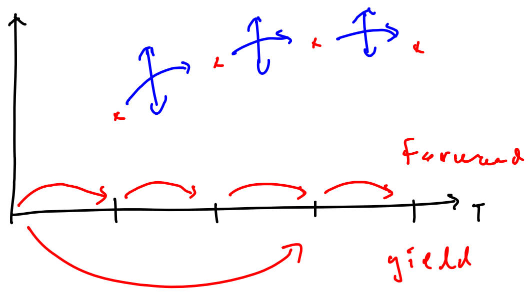
Market Models — BGM - I

IR

Bruce Gaterek, Musiela, Jamshidian



$$P_t(T_{k+1}) = (1 + l_t(T_k, T_{k+1})\,\Delta T_k)^{-1}\, P_t(T_k)$$

$$l_t^{(k)} = \frac{1}{\Delta T_k}\left(\frac{P_t^{(k)}}{P_t^{(k+1)}} - 1\right)$$

clearly $l_t^{(k)}$ is a $\mathbb{Q}^{(k+1)}$ mtg

($\mathbb{Q}^{(k+1)}$ is measure induced by $P_t^{(k+1)}$ as numeraire)

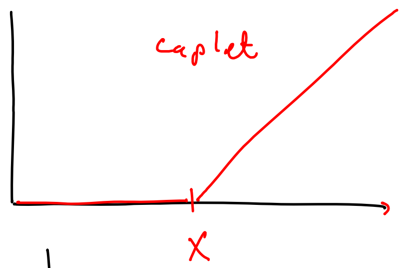
$$\frac{d\,l_t^{(k)}}{l_t^{(k)}} = \sigma_t^{(k)}\, dW_t^{(k+1)}$$

LFM - lognormal forward rate model

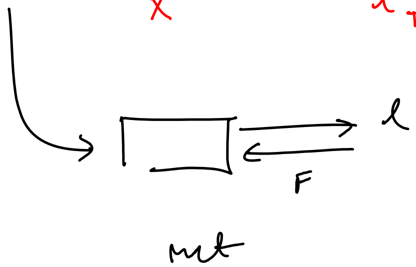
$\sigma_t^{(k)} \to$ const. / deterministic

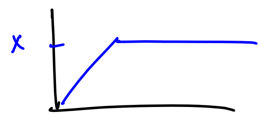
then $l_t^{(k)}\Big|_{\mathcal{F}_0} \sim$ log normal

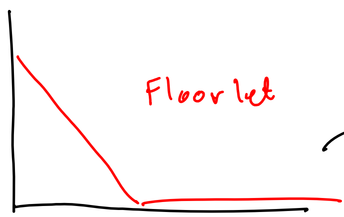
$\Rightarrow$ caps / floors $\longleftarrow$ caplets / floorlets

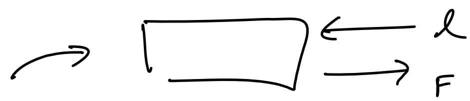


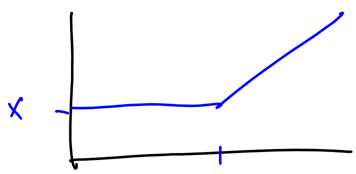
$$\ell_{T_k}(T_k, T_{k+1}) = \ell^{(k)}_{T_k} \quad @ \; T_{k+1}$$



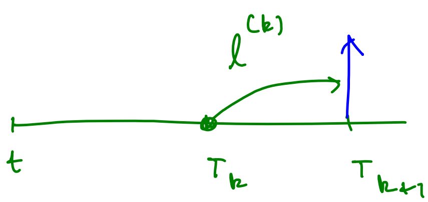








$$Cpl^{(k)}_t = \mathbb{E}^{\mathbb{Q}}_t\left[\left(\ell^{(k)}_{T_k} - X\right)_+ e^{-\int_t^{T_{k+1}} r_s \, ds} \right] N \, \Delta T_k$$



$\ell^{(k)}_t$ and under LFM it is lognormal

$$\frac{Cpl^{(k)}_t}{p^{(k+1)}_t} = \mathbb{E}^{\mathbb{Q}^{(k+1)}}_t\left[\frac{\left(\ell^{(k)}_{T_k} - X\right)_+}{p^{(k+1)}_{T_{k+1}}} \right]$$

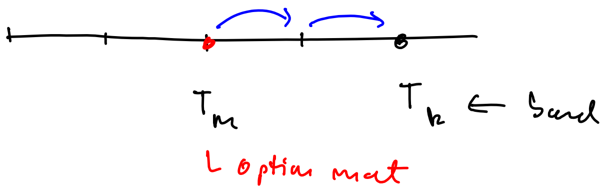
$\hookrightarrow 1$

$$= \mathbb{E}^{\mathbb{Q}^{(k+1)}}_t\left[\left(\ell^{(k)}_{T_k} - X\right)_+ \right]$$

$$= \ell_t^{(k)} \, \Phi(d_+) - X \, \Phi(d_-)$$

$$d_\pm = \frac{\ln\left(\ell_t^{(k)} / X\right) \pm \sigma_{eff}^2}{\sigma_{eff}}$$

$$\sigma_{eff}^2 = \int_t^T \left(\sigma_s^{(k)}\right)^2 ds$$



$$V_t = \mathbb{E}^{Q}\left[(P^{(k)}_{T_m} - X)_+ \, e^{-\int_t^{T_m} r_s \, ds} \right]$$

$$= P_t^{(m)} \, \mathbb{E}^{Q_m}\left[\frac{(P^{(k)}_{T_m} - X)_+}{P^{(m)}_{T_m}} \right]$$

$\hookrightarrow 1$

$$P^{(k)}_{T_m} / P^{(m)}_{T_m} \longrightarrow Y_t = \frac{P_t^{(k)}}{P_t^{(m)}}$$

$$Y_t = (1 + l_t^{(m)} \Delta T_m)^{-1} (1 + l_t^{(m+1)} \Delta T_{m+1})^{-1}$$
$$\cdots (1 + l_t^{(k-1)} \Delta T_{k-1})^{-1}$$

$$Q^{(k-1)} \longleftarrow Q^{(k)} \longleftarrow Q^{(k+1)}$$

$$\left(\frac{dQ^{(k)}}{dQ^{(k+1)}}\right)_t \text{"=" } \frac{dQ^{(k)}}{dQ} \cdot \frac{dQ}{dQ^{(k+1)}} = \frac{P_t^{(k)}}{P_t^{(k+1)}} \cdot c = \eta_t$$

$$\eta_t = c \cdot (1 + \Delta T_k \, l_t^{(k)})$$

$$\frac{d\eta_t}{\eta_t} = c \, \Delta T_k \frac{d l_t^{(k)}}{\eta_t} \qquad \left(\frac{d l_t^{(k)}}{l_t^{(k)}} = \sigma_t^{(k)} \, dW_t^{(k+1)} \right)$$

$$= \frac{c \, \Delta T_k \, l_t^{(k)} \, \sigma_t^{(k)}}{\eta_t} \, dW_t^{(k+1)}$$

$$= \frac{\Delta T_k \, l_t^{(k)}}{} \cdot \sigma_t^{(k)} \, dW_t^{(k+1)}$$

$$1 + \Delta T_k \ell_t^{(k)}$$

Girsanov

$\Rightarrow$

$$\boxed{dW_t^{(k)} = -\frac{\Delta T_k \ell_t^{(k)}}{1 + \Delta T_k \ell_t^{(k)}} \cdot \sigma_t^{(k)} dt + dW_t^{(k+1)}}$$

is a $\mathbb{Q}^{(k)}$ - B. mtn.

so $$\frac{d\ell^{(k)}}{\ell^{(k)}} = \sigma_t^{(k)} dW_t^{(k+1)}$$

$$= \frac{\Delta T_k \ell_t^{(k)}}{1 + \Delta T_k \ell_t^{(k)}} \cdot (\sigma_t^{(k)})^2 dt + \sigma_t^{(k)} dW_t^{(k)}$$

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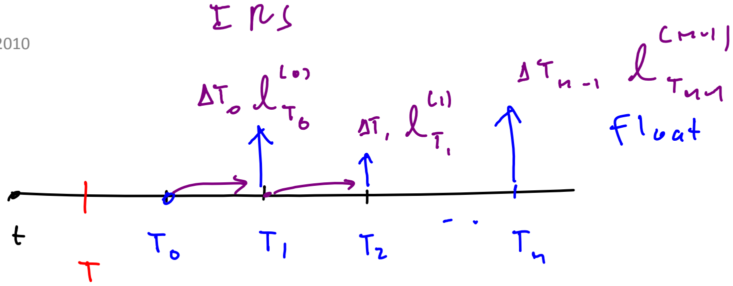
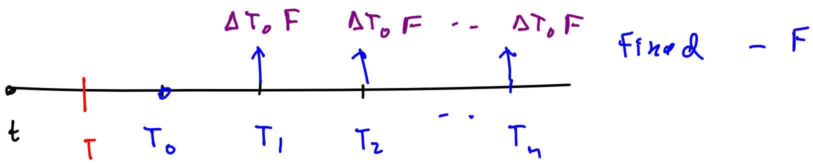
IRS



ΔT all equal..



$$V_t^{fix} = \Delta T \, F \sum_{k=1}^{n} P_t(T_k)$$

$$V_t^{fl} = P_t(T_0) - P_t(T_n)$$

$$\Rightarrow F_t = \frac{P_t(T_0) - P_t(T_n)}{\Delta T \sum_{k=1}^{n} P_t(T_k)}$$

Payer swaption (swap option) has right to enter into payer leg of swap at T at a rate K.

option payoff

$$\varphi = \mathbb{1}_{F_T > K} \left(V_T^{fl} - V_T^{fix} \right)$$

$$= \mathbb{1}_{F_T > K} \left(\left(P_T(T_0) - P_T(T_n) \right) - \Delta T \, K \sum_{k=1}^{n} P_T(T_k) \right)$$

$$= \left(\Delta T \sum_{k=1}^{n} P_T(T_k) \right) \mathbb{1}_{F_T > K} \left(\frac{P_T(T_0) - P_T(T_n)}{\Delta T \sum_{k=1}^{n} P_T(T_k)} - K \right)$$

$$= \underbrace{\left(\Delta T \sum_{k=1}^{n} P_T(T_k) \right)}_{\rightsquigarrow A_t \text{ annuity}} \left(F_T - K \right)_+$$

$$V_t = \mathbb{E}_t^{\mathbb{Q}} \left[\varphi \, e^{-\int_t^T r_s \, ds} \right]$$

$$= A_t \, \mathbb{E}_t^{\mathbb{Q}^A}\left[\frac{\mathbb{Q}}{A_T} \right]$$

$$= A_t \, \mathbb{E}_t^{\mathbb{Q}^A}\left[(F_T - K)_+ \right]$$

and F_t is a $\mathbb{Q}^A$ - mtg !!!

$$\frac{dF_t}{F_t} = \sigma_t^F \, dW_t^A$$

if deterministic then $F_T \sim$ log normal

LSM
log-normal swap rate model

$\Rightarrow$ Black-like formulas