Bertsimas & Lo (discrete)

13 March 2013 14:10

Almgren & Chriss (discrete → continuous)

$$dQ_t^\nu = -\nu_t \, dt$$

rate of trading

$$dS_t = \sigma \, dW_t$$ Fundamental or mid-price

$$\hat{S}_t^\nu = S_t - \underbrace{a \nu_t}_{\text{temporary impact}}$$ execution price.

$$dX_t^\nu = \hat{S}_t^\nu \, \nu_t \, dt$$ cash process

$$V^\nu = X_T^\nu + \underbrace{Q_T^\nu (S_T - \alpha Q_T^\nu)}_{\text{value of terminal liquidation}}$$ total wealth at T

$$H(x,q,S) = \sup_{\nu \in \mathcal{A}} \mathbb{E}[X_T^\nu + Q_T^\nu (S_T - \alpha Q_T^\nu) \mid X_0 = x, Q_0 = q, S_0 = S]$$

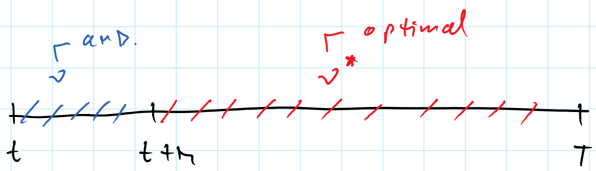
performance criteria

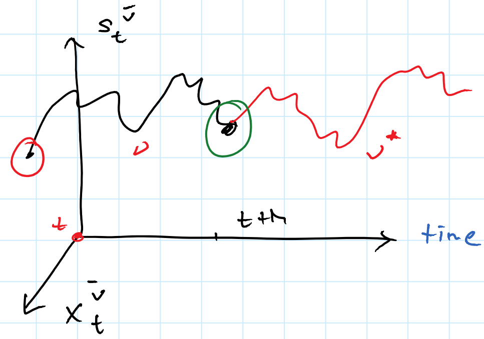
$$H^\nu(t,x,q,S) = \mathbb{E}[X_T^\nu + Q_T^\nu (S_T - \alpha Q_T^\nu) \mid X_t = x, Q_t = q, S_t = S]$$

$$H(t,x,q,S) = \sup_{\nu \in \mathcal{A}} H^\nu(t,x,q,S)$$

value Fn.

$\tilde{\nu}$





$$H^{\tilde{\nu}}(\cdot) = \mathbb{E}[V^{\tilde{\nu}} \mid X_t = x, Q_t = q, S_t = S]$$

$$= \mathbb{E}[\underbrace{\mathbb{E}[V^{\tilde{\nu}} \mid \mathcal{F}_{t+h}]}_{H^{\nu^*}(t+h,\, X_{t+h}^{\nu},\, Q_{t+h}^{\nu},\, S_{t+h}^{\nu})} \mid \cdot]$$

$\tilde{\nu}$

$$H^{\tilde{\nu}}(t,x,q,S) = \mathbb{E}[H^{\nu^*}(t+h, X_{t+h}^\nu, Q_{t+h}^\nu, S_{t+h}^\nu) \mid X_t = x, Q_t = q, S_t = S]$$

① $$H(t+h, W_{t+h}) = H(t, W_t) + \int_t^{t+h} (\partial_t + \tfrac{1}{2}\partial_{ww}) H(s, W_s) \, ds$$

① $$H(t+h, W_{t+h}) = H(t, W_t) + \int_t^{\cdot} \left(\partial_t + \tfrac{1}{2}\partial_{ww}\right) H(s, W_s)\, ds$$
$$+ \int_t^{t+h} \partial_w H(s, W_s)\, dW_s$$

$$d\,H(t, W_t) = \left(\partial_t + \tfrac{1}{2}\partial_{ww}\right) H(t, W_t)\, dt + \partial_w H(t, W_t)\, dW_t$$

② $$dY_t = \mu(t, Y_t)\, dt + \sigma(t, Y_t)\, dW_t$$
$$H(t+h, Y_{t+h}) = H(t, Y_t) + \int_t^{t+h} \left(\partial_t + \mu(s, Y_s)\,\partial_y + \tfrac{1}{2}\sigma^2(s, Y_s)\,\partial_{yy}\right) H(s, Y_s)\, ds$$
$$+ \int_t^{t+h} \sigma(s, Y_s)\,\partial_y H(s, Y_s)\, dW_s$$

$$H(t+h, Y_{t+h}, W_{t+h}) = H(t, Y_t, W_t)$$
$$+ \int_t^{t+h} \Big\{ \partial_t + \mu(s, Y_s)\,\partial_y + 0\,\partial_w$$
$$+ \tfrac{1}{2}\sigma^2(s, Y_s)\,\partial_{yy} + \tfrac{1}{2}\partial_{ww} + \sigma(s, Y_s)\,\partial_{wy} \Big\} H(s, Y_s, W_s)\, ds$$
$$+ \int_t^{t+h} \sigma(s, Y_s)\,\partial_y H(s, Y_s, W_s)\, dW_s + \int_t^{t+h} \partial_w H(s, Y_s, W_s)\, dW_s$$

$$H(t+h, X^\nu_{t+h}, Q^\nu_{t+h}, S^\nu_{t+h}) = H(t, X_t, Q_t, S_t)$$
$$+ \int_t^{t+h} \left(\partial_t + \hat{S}_s \nu_s \partial_x - \nu_s \partial_q + \tfrac{1}{2}\sigma^2 \partial_{ss}\right) H(s, X^\nu_s, Q^\nu_s, S^\nu_s)\, ds$$
$$+ \int_t^{t+h} \sigma\, \partial_s H(s, X^\nu_s, Q^\nu_s, S^\nu_s)\, dW_s$$

$$\Rightarrow\ H^\nu(t, x, q, S) = H(t, x, q, S)$$
$$+ \mathbb{E}_{t,x,q,s}\left[\int_t^{t+h} \left(\partial_t + (S_s - a\nu_s)\nu_s \partial_x - \nu_s \partial_q + \tfrac{1}{2}\sigma^2 \partial_{ss}\right) H(s, X^\nu_s, Q^\nu_s, S^\nu_s)\, ds\right]$$

take $\sup_{\nu \in \mathcal{A}[t, t+h]} \Big[(\quad) = (\quad) \Big]$

$$\Rightarrow\ H(t, x, q, S) = H(t, x, q, S) + \sup_{\nu \in \mathcal{A}[t,t+h]} \mathbb{E}_{t,x,q,s}[\ \cdot\ ]$$

$$\Rightarrow\ 0 = \sup_{\nu \in \mathcal{A}[t,t+h]} \mathbb{E}_{t,x,q,s}\left[\frac{1}{h}\int_t^{t+h} \mathcal{H}_s\, ds\right]$$

$h \downarrow 0$

$$\Rightarrow \quad 0 = \sup_{\nu \in \mathcal{A}_t} \mathbb{E}_{t,x,q,s}\left[\mathcal{H}_t \right]$$

$$= \sup_{\nu \in \mathcal{A}_t} \Big(\partial_t + \underbrace{(S - a\nu)\,\nu\,\partial_x - \nu\,\partial_q + \tfrac{1}{2}\sigma^2 \partial_{ss}}_{\mathcal{L}^{\nu, X_t, Q_t, S_t}} \Big) H(t,x,q,s)$$

subject to:
$H(T,x,q,s) = x + q(S - \alpha q)$

This is the HJB equation (DPE)

$$H(t,x,q,S) = \sup_{\nu \in \mathcal{A}} \mathbb{E}_{t,x,q,S}\left[X_T^{\nu} + Q_T^{\nu}\left(S_T - \alpha Q_T^{\nu}\right)\right]$$

$$\begin{cases} \partial_t H + \sup_{\nu \in \mathcal{A}_t} \Big(\overbrace{(S - a\nu)\nu\, \partial_x - \nu\, \partial_q + \tfrac{\sigma^2}{2}\partial_{SS}}^{\mathcal{L}^{\nu}} \Big) H = 0 \\ H(T,x,q,S) = x + q(S - \alpha q) \end{cases}$$

$$\nu\left(S\,\partial_x H - \partial_q H\right) - a\nu^2\, \partial_x H$$

$$= -a\,\partial_x H \left[\nu^2 - \nu\, \frac{S\,\partial_x H - \partial_q H}{a\,\partial_x H} \right]$$

$$= -a\,\partial_x H \left[\left(\nu - \frac{S\,\partial_x H - \partial_q H}{2a\,\partial_x H}\right)^2 - \left(\frac{S\,\partial_x H - \partial_q H}{2a\,\partial_x H}\right)^2 \right]$$

$$\nu^* = \frac{S\,\partial_x H - \partial_q H}{2a\,\partial_x H}$$

$$\sup(\quad) = \frac{\left(S\,\partial_x H - \partial_q H\right)^2}{4a\,\partial_x H}$$

$$\Rightarrow \begin{cases} \partial_t H + \dfrac{\left(S\,\partial_x H - \partial_q H\right)^2}{4a\,\partial_x H} + \tfrac{1}{2}\sigma^2 \partial_{SS} H = 0 \\ H(T,x,q,S) = x + q(S - \alpha q) \end{cases}$$

$$H(t,x,q,S) = x + qS + h(t,q)$$

$$h(T,q) = -\alpha q^2$$

$$\partial_t h + \frac{\left(S - (S + \partial_q h)\right)^2}{4a} + 0 = 0$$

$$\Rightarrow \partial_t h + \frac{(\partial_q h)^2}{4a} = 0$$

$$h(t,q) = \eta(t)(-\alpha q^2), \qquad \eta(T) = 1$$

$$-\alpha q^2\, \partial_t \eta + \frac{(-2\alpha q\, \eta)^2}{4a} = 0$$

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$$(-\alpha q^2)\Big(\underbrace{\partial_t \eta - \frac{\alpha}{a} \eta^2}_{=0} \Big) = 0$$

$$\begin{cases} \dot{\eta} - \frac{\alpha}{a} \eta^2 = 0 \\ \eta(T) = 1 \end{cases}$$

$$\frac{\dot{\eta}}{\eta^2} = \frac{\alpha}{a} \Rightarrow -\Big(\underbrace{\frac{1}{\eta(T)}}_{\hookrightarrow 1} - \frac{1}{\eta(t)} \Big) = \frac{\alpha}{a}(T-t)$$

$$\Rightarrow \qquad \eta(t) = \Big(1 + \frac{\alpha}{a}(T-t)\Big)^{-1}$$

$$H(t,x,q,S) = x + qS - \alpha q^2 \Big(1 + \frac{\alpha}{a}(T-t)\Big)^{-1}$$

$$\nu^* = \frac{S\,\partial_x H - \partial_q H}{2a\,\partial_x H} = \frac{S - (S - 2\alpha q\,\eta(t))}{2a}$$

$$= \frac{\alpha}{a} q \Big(1 + \frac{\alpha}{a}(T-t)\Big)^{-1}$$

$$= q\Big(\frac{a}{\alpha} + (T-t)\Big)^{-1}$$

$$dQ_t = -\nu_t^* \, dt = -Q_t \Big(\frac{a}{\alpha} + (T-t)\Big)^{-1} dt$$

$$\frac{dQ_t}{Q_t} = -\Big(\frac{a}{\alpha} + (T-t)\Big)^{-1} dt$$

$$\ln Q_t - \ln Q_0 = -\int_0^t \Big(\frac{a}{\alpha} + (T-s)\Big)^{-1} ds$$

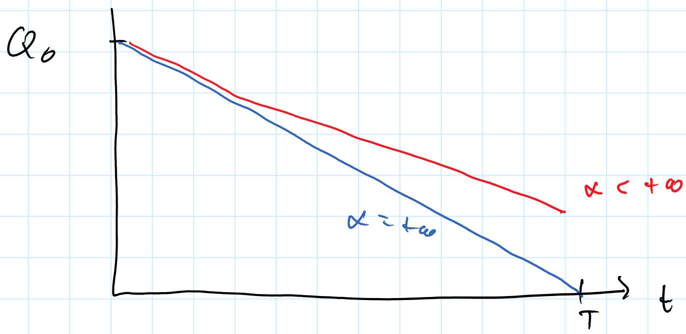
$$= -\int_0^t \Big(\Big(\frac{a}{\alpha} + T\Big) - s\Big)^{-1} ds$$

$$= \ln \frac{\Big(\Big(\frac{a}{\alpha} + T\Big) - t\Big)}{\Big(\frac{a}{\alpha} + T\Big)}$$

$$= \ln\left(1 - \frac{t}{\left(\frac{a}{\alpha} + T\right)}\right)$$

$$\Rightarrow \boxed{Q_t = Q_0 \left(1 - \frac{t}{\frac{a}{\alpha} + T}\right)}$$

$$\boxed{v_t = \frac{Q_0}{\frac{a}{\alpha} + T}}$$



$$\mathbb{E}[X_T + Q_T(S_T - \alpha Q_T)]$$

$$\to \mathbb{E}[u(X_T + Q_T(S_T - \alpha Q_T))]$$ utility on terminal wealth.

$$\to \mathbb{E}\left[X_T + Q_T(S_T - \alpha Q_T) - \phi \int_0^T Q_s^2\, ds\right]$$

$$\int_0^T \left(Q_s - \left(1 - \frac{s}{T}\right) Q_0\right)^2 ds$$

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① $Y_t = Q_t S_t$ book value of inventory

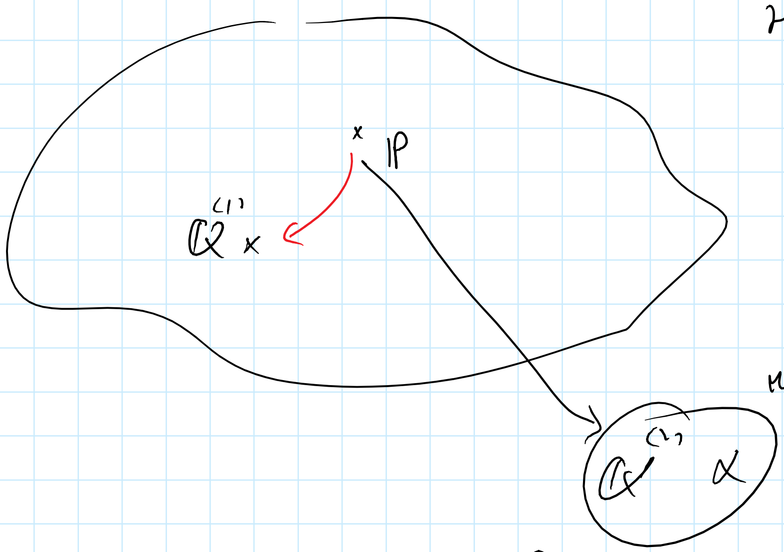
$[Y, Y]_t = \ldots$

$$dY_t = \underbrace{dQ_t}_{\hookrightarrow\, -\nu_t\, dt} S_t + Q_t \underbrace{dS_t}_{\hookrightarrow\, \sigma\, dW_t} + \underbrace{d[Q, S]_t}_{\leftarrow\, 0}$$

$$[Y, Y]_t = \sigma^2 \int_0^t Q_s^2\, ds$$

② $$dS_t = \sigma\, dW_t$$

$$\mathcal{H}(\mathbb{Q} \,|\, \mathbb{P}) = \mathbb{E}^{\mathbb{Q}}\left[\ln \frac{d\mathbb{Q}}{d\mathbb{P}}\right]$$ relative entropy

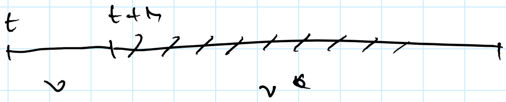


no weight in your decision

$$\inf_{\mathbb{Q} \in \mathcal{Q}} \sup_{\nu \in \mathcal{A}} \mathbb{E}^{\mathbb{Q}}\left[X_T + Q_T(S_T - \alpha Q_T) - \phi \ln \frac{d\mathbb{Q}}{d\mathbb{P}}\right]$$

Ambiguity Aversion

$$H(t,x,q,S) = \sup_{\nu \in \mathcal{A}} \mathbb{E}_{t,x,q,S}\Big[\overbrace{X_T^{\nu} + Q_T^{\nu}\left(S_T^{\nu} - \alpha Q_T^{\nu}\right)}^{V^{\nu}} - \phi \int_t^T (Q_s^{\nu})^2\, ds \Big]$$



$$H^{\tilde\nu} = \mathbb{E}_{t,x,q,S}\Big[\mathbb{E}\big[V^{\tilde\nu} - \phi \underbrace{\int_t^T (Q_s^{\tilde\nu})^2\, ds}_{\int_t^{t+h} \cdot \; + \; \int_{t+h}^T \cdot} \,\big|\, \mathcal{F}_{t+h} \big]\Big]$$

$$= \mathbb{E}_{t,x,q,S}\Big[\mathbb{E}\big[V^{\tilde\nu} - \phi \int_{t+h}^T (Q_s^{\tilde\nu})^2\, ds \,\big|\, \mathcal{F}_{t+h}\big]$$

↳ $H(t+h, X^{\nu}_{t+h}, Q^{\nu}_{t+h}, S^{\nu}_{t+h})$

↳ $\int_t^{t+h} (\partial_t + \mathcal{L}^{\nu}) H_s\, ds$

$$- \phi \int_t^{t+h} (Q_s^{\tilde\nu})^2\, ds \Big]$$

↳ Q_s^{ν}

take $\sup_{\nu}$, $\frac{1}{h}$, $\lim_{h \downarrow 0}$ ...

$$\Rightarrow \quad 0 = \sup_{\nu \in \mathcal{A}_t} \left((\partial_t + \mathcal{L}^{\nu}) H - \phi q^2 \right)$$

$$H(T,x,q,S) = x + q(S - \alpha q)$$

$$H(t,y) = \sup_{\nu \in \mathcal{A}} \mathbb{E}\Big[G(Y_T^{\nu}) + \int_t^T F(s, Y_s^{\nu})\, ds \Big]$$

$$\begin{cases} \partial_t H + \sup_{\nu} (\mathcal{L}^{\nu} H) + F(t,y) = 0 \\ H(T,y) = G(y) \end{cases}$$

DPE.

$$dX_t = \kappa(\theta - X_t)\, dt + \sigma\, dW_t$$

$$[X,X]_t = \sigma^2 t$$

$$\mathbb{V}[X_T] = \mathbb{E}\Big[\int_0^T \left(e^{-\kappa(T-s)}\right)^2 ds \Big] \sigma^2$$

$$\mathbb{V}[X_T] = \mathbb{E}\left[\int_0^T \left(e^{-\kappa(T-s)}\right)^2 ds\right] \sigma^2$$

$$X_T = X_0 e^{-\kappa(T-t)} + \theta(1 - e^{-\kappa(T-t)}) + \underbrace{\sigma \int_t^T e^{-\kappa(T-s)} dW_s}$$

$$\Rightarrow \mathbb{V}[X_t] = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t}) \xrightarrow[t \to +\infty]{} \frac{\sigma^2}{2\kappa}$$

$$dS_t = \sigma dW_t - \underbrace{b v_t dt}_{\text{permanent impact.}}$$