Random Walk

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$$S \begin{cases} S e^{\sigma\sqrt{\Delta t}} \\ S e^{-\sigma\sqrt{\Delta t}} \end{cases} \qquad\qquad r \begin{cases} r + \sigma\sqrt{\Delta t} \\ r - \sigma\sqrt{\Delta t} \end{cases}$$

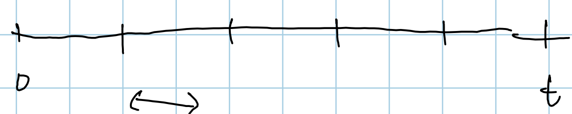
$$S e^{\sigma W} \qquad\qquad\qquad r + \sigma W$$

$$0 \begin{cases} +\sqrt{\Delta t} \\ -\sqrt{\Delta t} \end{cases} \qquad W \longrightarrow \text{Brownian motion}$$

$$W_{n\Delta t} = W_{(n-1)\Delta t} + \sqrt{\Delta t}\, x_n .$$

$x_1, x_2, \ldots$ are iid Bernoulli

$$\mathbb{P}(x_k = \pm 1) = 1/2$$



$\Delta t = t/N$

$$W_0 = 0$$

$$W_t \underset{N\to\infty}{\overset{d}{=}} Z\sqrt{t}, \qquad Z \underset{\mathbb{P}}{\sim} N(0,1)$$

$$W_{N\Delta t} = \sqrt{\Delta t} \sum_{m=1}^{N} x_m$$

$$\mathbb{E}[W_{N\Delta t}] = \sqrt{\Delta t} \sum \mathbb{E}[x_m] = 0$$

$$\mathbb{V}[W_{N\Delta t}] = \Delta t \sum \mathbb{V}[x_m] = \Delta t \sum \left[\mathbb{E}[x_m^2] - (\mathbb{E}[x_m])^2 \right]$$

$$= \Delta t \sum (1 - 0) = \Delta t N = t$$

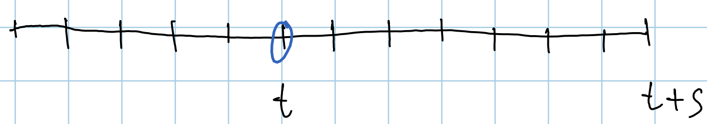
By CLT $\quad W_t \xrightarrow[N \to \infty]{d} \sqrt{t}\, Z \quad , \quad Z \underset{\mathbb{P}}{\sim} N(0,1)$

by CLT

$y_1 \; y_2 \; \cdots \; y_M \longrightarrow$ iid Bernoulli $\quad (y_k = x_{N+k})$

$\mathbb{P}(y_k = \pm 1) = \frac{1}{2}$

$x_1 \; x_2 \; \cdots \; x_N \; x_{N+1} \; \cdots \; x_{N+M}$



$$W_{t+s} - W_t = \sum_{n=1}^{N+M} x_n \sqrt{\Delta t} - \sum_{n=1}^{N} x_n \sqrt{\Delta t}$$

$$= \sum_{n=N+1}^{N+M} x_n \sqrt{\Delta t}$$

$$= \sum_{n=1}^{M} y_n \sqrt{\Delta t} \quad , \quad M \Delta t = s$$

$$\text{by CLT} \quad \xrightarrow[M \to +\infty]{d} \sqrt{s}\, Z \, , \quad Z \underset{\mathbb{P}}{\sim} N(0,1)$$

since $W_{t+s} - W_t$ does not depende on W_t (or t) then W has stationary increments.

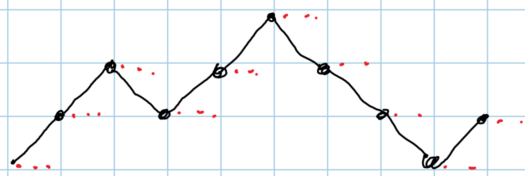
How about joint distribution of

$$W_t \, , \; (W_{t+s} - W_t) \quad ?$$

are independent (even at finite Δt)

$(W_s - W_t) \perp (W_v - W_u)$

(independent increments)



W has continuous paths

Brownian Motion & Total Variation

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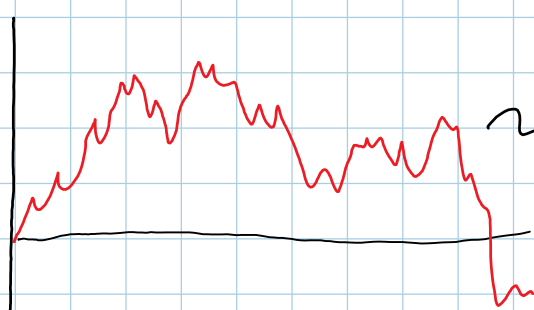
- $W_0 = 0$
- $W_t \overset{d}{=} \sqrt{t}\, Z, \quad Z \sim N(0,1)$
- W_t has stationary & independent increments.
- W_t has continuous paths.

Such a process is called a Brownian motion.



~ "Hairy paths"

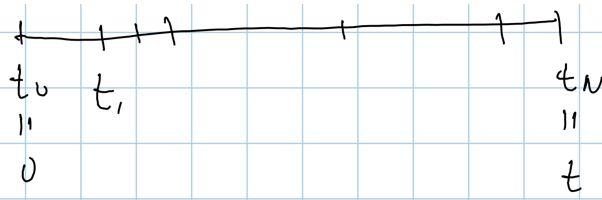
· nowhere differentiable.

Total Variation:

$f(t)$, TV is defined as:

$$TV_t = \lim_{\|\Pi\| \downarrow 0} \sum_k \underbrace{\left| f(t_k) - f(t_{k-1}) \right|}$$

$$\Pi = \{ t_0, t_1, \ldots, t_N \} \quad 0 = t_0 < t_1 < \cdots < t_N = t$$

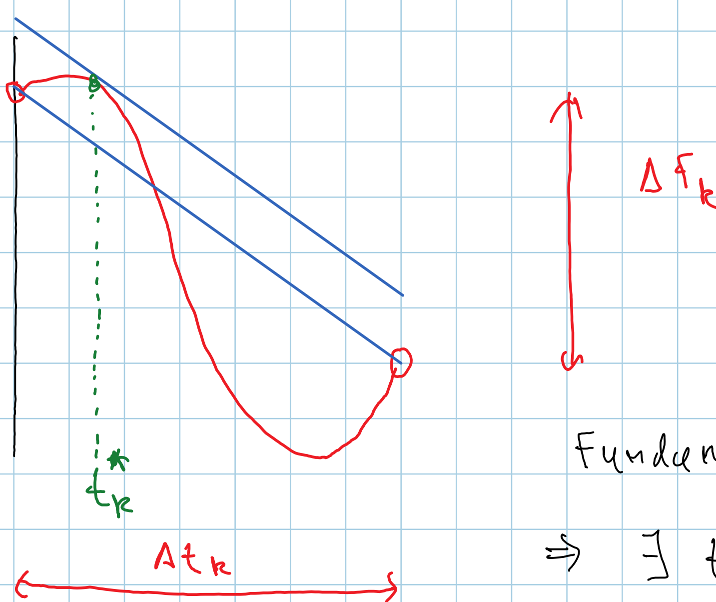


$$\|\Pi\| = \max_k \{ (t_k - t_{k-1}) \}.$$

$$|f(t_1) - f(t_0)| + |f(t_2) - f(t_1)| = 2$$

① f is differentiable:

fix Π. $\quad \sum_k |f(t_k) - f(t_{k-1})|$



Fundamental thm of calc

$\Rightarrow \exists\, t_k^* \in (t_{k-1}, t_k)$

s.t. $f'(t_k^*) = \dfrac{\Delta f_k}{\Delta t_k}$

$$\Rightarrow \quad \Delta f_k = f'(t_k^*)\, \Delta t_k$$

$$\text{so} \quad \sum_k |\Delta f_k| = \sum_k |f'(t_k^*)|\, \Delta t_k$$

$$\lim_{\|\Pi\| \downarrow 0} \sum_k |f'(t_k^*)| \Delta t_k = \int_0^t |f'(s)| \, ds < +\infty$$

(2) Brownian case: given Π

$$\sum_k | \underbrace{W_{t_k} - W_{t_{k-1}}}_{\Delta W_k \overset{d}{=} (\Delta t_k)^{1/2} Z_k} | \xrightarrow[\|\Pi\| \downarrow 0]{} +\infty \ !!!$$

$Z_1, Z_2, \ldots$ iid $N(0,1)$

$$= \sum_k (\Delta t_k)^{1/2} |Z_k|$$

$$\sum_k \Delta t_k |Z_k| \xrightarrow[\|\Pi\| \downarrow 0]{} t \, \mathbb{E}[|Z|] \quad a.s.$$

L.L.N. $\quad \mathbb{E}[\cdot] \to \#$

$\mathbb{V}[\cdot] \xrightarrow[\|\Pi\| \downarrow 0]{} 0$

$$\mathbb{E}\Big[\sum_k \Delta t_k |Z_k|\Big] = \sum_k \Delta t_k \underbrace{\mathbb{E}[|Z_k|]}_{\mathbb{E}[|Z|]} = \mathbb{E}[|Z|] \cdot t$$

$$\mathbb{V}\Big[\sum_k \Delta t_k |Z_k|\Big] = \sum_k \Delta t_k^2 \underbrace{\mathbb{V}[|Z_k|]}_{c}$$

$$= c \sum_k \Delta t_k^2$$

$$\leq c \sum_k \|\pi\| \Delta t_k$$

$$= c \|\pi\| \underbrace{\sum_k \Delta t_k}_{t} \xrightarrow[\|\pi\| \downarrow 0]{} 0$$

$$\sum_k (\Delta t_k)^{1/2} |Z_k| \geq \sum_k \frac{\Delta t_k}{\|\pi\|^{1/2}} |Z_k|$$

$$= \frac{1}{\|\pi\|^{1/2}} \underbrace{\sum_k \Delta t_k |Z_k|}_{\to\ t\, \mathbb{E}[|Z|]}$$

$$\xrightarrow[\|\pi\| \downarrow 0]{} +\infty$$

Quadratic Variation

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Quadratic Variation:

$$[f]_t = [f,f]_t = \lim_{\|\Pi\| \downarrow 0} \sum_k \Delta f_k^2$$

① f is differentiable

$$\sum_k \Delta f_k^2 = \sum_k \left(f'(t_k^*)\right)^2 \Delta t_k^2$$

$\hookrightarrow \Delta t_k \Delta t_k \leq \Delta t_k \|\Pi\|$

$$\leq \left(\sum_k (f'(t_k^*))^2 \Delta t_k \right) \|\Pi\|$$

$\hookrightarrow \int_0^t (f'(s))^2 \, ds < +\infty$

$$\xrightarrow[\|\Pi\| \downarrow 0]{} 0$$

② B. mtn.

$$Q = \sum_k \Delta W_k^2$$

$$\mathbb{E}[Q] = \sum_k \mathbb{E}[\Delta W_k^2] = \sum_k \Delta t_k = t$$

$$\mathbb{V}[Q] = \sum_k \mathbb{V}[\Delta W_k^2] = \sum_k \Delta t_k^2 \, \mathbb{V}[Z^2]$$

$$\hookrightarrow \mathbb{V}[(\Delta t_k^{1/2} Z)^2]$$

$$= \mathbb{V}[\Delta t_k Z^2] = \Delta t_k^2 \mathbb{V}[Z^2]$$

$$\leq \mathbb{V}[Z^2] \cdot \|\Pi\| \underbrace{\sum_k \Delta t_k}_{t}$$

$$\xrightarrow[\|\Pi\| \downarrow 0]{} 0$$

$$\therefore \sum_k \Delta W_k^2 \xrightarrow[\|\Pi\| \downarrow 0]{} t \quad a.s.$$

$$\boxed{[W, W]_t = t \quad a.s.}$$

Stochastic Integral... First Steps

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$$\int_0^t \underbrace{f(s)\,df(s)}_{\frac{1}{2}\,d(f^2(s))} = \frac{1}{2}\left(f^2(t) - f^2(0)\right)$$

$$\int_0^t W_s\,dW_s = \lim_{\|\Pi\|\to 0} \sum_k W_{t_{k-1}}\,\Delta W_{t_k}$$

$$\overset{?}{=} \frac{1}{2}\left(W_t^2 - W_0^2\right) = \frac{1}{2} W_t^2 \;?$$

$$\frac{1}{2} W_t^2 - \int_0^t W_s\,dW_s = \frac{1}{2} t$$

$$\int_0^t W_s\,dW_s = \frac{1}{2}\left(W_t^2 - t\right)$$