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$$\int_0^t g(s, W_s)\, dW_s \triangleq \lim_{\|\Pi\| \downarrow 0} \sum_k g(t_{k-1}, W_{t_{k-1}})\, \Delta W_k$$

$(W_{t_k} - W_{t_{k-1}})$

$$\Pi = \{ t_0, t_1, \ldots, t_n \}$$

$$0 = t_0 < t_1 < \cdots < t_n = t$$

Ito's lemma I:

if $X_t = h(W_t)$, and $h(y)$ is twice diff.

then $dX_t = h'(W_t)\, dW_t + \underline{\tfrac{1}{2} h''(W_t)\, dt}$

Ito correction

stochastic differential equation

$$\left(X_t - X_0 = \int_0^t h'(W_s)\, dW_s + \frac{1}{2} \int_0^t h''(W_s)\, ds \right)$$

e.g. $X_t = W_t^2$, $h(y) = y^2$

$$dX_t = 2 W_t\, dW_t + \frac{1}{2} \cdot 2 \cdot dt$$

$$= 2 W_t\, dW_t + dt$$

$$X_t - X_0 = 2 \underline{\int_0^t W_s\, dW_s} + \int_0^t ds$$

$$\Rightarrow \int_0^t W_s\, dW_s = \frac{1}{2}\left[(W_t^2 - W_0^2) - t \right]$$

$$= \frac{1}{2} \left[W_t^2 - t \right]$$

Ito correction.

e.g. int. by. parts for $\int_0^t W_s^4 \, dW_s$

natural to consider $X_t = W_t^5$, $h(y) = y^5$

$$dX_t = \underbrace{5 W_t^4 \cdot dW_t}_{h'(W_t)\, dW_t} + \underbrace{\tfrac{1}{2} \cdot 5 \cdot 4 \cdot W_t^3 \, dt}_{\frac{1}{2} h''(W_t)\, dt}$$

$$\Rightarrow \int_0^t W_s^4 \, dW_s = \frac{1}{5} \left[W_t^5 - 10 \int_0^t W_s^3 \, ds \right]$$

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suppose that S_t satisfies the SDE:

$$dS_t = \sigma S_t \, dW_t$$

What is the solution this SDE?

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reminder: solve the ODE:

$$dz_t = \sigma z_t \, dg_t$$

$$z_t = (\sigma, g, \int, g' \ldots)$$

$$\frac{dz_t}{z_t} = \sigma \, dg_t$$

$$d(\ln z_t) = \frac{dz_t}{z_t} = \sigma \, dg_t$$

$$\Rightarrow \ln z_t - \ln z_0 = \sigma (g_t - g_0)$$

$$\Rightarrow \boxed{z_t = z_0 \, e^{\sigma (g_t - g_0)}}$$

$$\frac{dS_t}{S_t} = \sigma \, dW_t \iff dS_t = \sigma S_t \, dW_t$$

$$d \underbrace{\ln S_t}_{h(S_t)} \neq \sigma \, dW_t$$

where $h(y) = \ln y$

Ito's lemma II:

suppose $dY_t = \sigma(t, Y_t) \, dW_t$ and

$X_t = h(Y_t)$ with $h(y)$ being twice diff, then

$$dX_t = h'(Y_t) \underbrace{dY_t}_{\hookrightarrow \sigma(t, Y_t)\, dW_t} + \underbrace{\tfrac{1}{2} h''(Y_t) \cdot \sigma^2(t, Y_t) \, dt}_{\hookrightarrow \text{Ito correction.}}$$

$X_t = h(W_t)$

$$dX_t = h'(W_t) \, dW_t + \tfrac{1}{2} h''(W_t) \, dt$$

so $X_t = \ln S_t$, $h(y) = \ln y$

$$dX_t = \left(\frac{1}{S_t}\right) \underbrace{dS_t}_{\hookrightarrow \sigma S_t dW_t} + \frac{1}{2}\left(-\frac{1}{S_t^2}\right) \cdot (\sigma S_t)^2 \, dt$$

$$= \sigma \, dW_t \underbrace{- \tfrac{1}{2}\sigma^2 \, dt}_{\hookrightarrow \text{Ito correction}}$$

$$\Rightarrow X_t - X_0 = \sigma (W_t - W_0) - \tfrac{1}{2}\sigma^2 t$$

$$\Rightarrow X_t = X_0 + \sigma (W_t - W_0) - \tfrac{1}{2}\sigma^2 t$$

$$\Rightarrow \quad \ln S_t = \ln S_0 + \sigma(W_t - W_0) - \tfrac{1}{2}\sigma^2 t$$

$$S_t = S_0 \, e^{\sigma(W_t - W_0) - \frac{1}{2}\sigma^2 t} \iff dS_t = \sigma S_t \, dW_t$$

$$Z_t = Z_0 \, e^{\sigma(g_t - g_0)} \iff dZ_t = \sigma Z_t \, dg_t$$

$$\frac{dS_t}{S_t} = \sigma \, dW_t$$

instantaneous return of S_t

dW_t — source of noise

σ — size of fluctuations (volatility)

μ — drift or expected instantaneous return

$$\frac{dS_t}{S_t} = \mu \, dt + \sigma \, dW_t$$

Ito lemma's:

suppose that Y_t satisfies the SDE

$$dY_t = \mu(t, Y_t)\, dt + \sigma(t, Y_t)\, dW_t$$

and $X_t = h(t, Y_t)$ which is once diff in t
& twice " " y

then:

$$dX_t = \partial_y h(t, Y_t)\, dY_t + \partial_t h(t, Y_t)\, dt$$

$$\underbrace{+ \tfrac{1}{2}\, \partial_{yy} h(t, Y_t)\, \sigma^2(t, Y_t)\, dt}_{\text{Ito correction}}$$

(note for B. mtn $\mu = 0$, $\sigma = 1$)

e.g. Find an integration by parts formula for:

$$\int_0^t s\, W_s \, dW_s$$

$\hookrightarrow \partial_y h(t, W_t)$ so choose $h(t,y) = t\, y^2$

$x_t = h(t, W_t)$

so $dX_t = 2t\, W_t \cdot dW_t + W_t^2\, dt$

$+ \frac{1}{2}\, 2t \cdot dt$

$$\Rightarrow \quad t\, W_t\, dW_t = \frac{1}{2}\left[dX_t - (W_t^2 - t)\, dt \right]$$

$$\Rightarrow \quad \int_0^t s\, W_s\, dW_s = \frac{1}{2}\left[X_t - X_0 - \int_0^t (W_s^2 - s)\, ds \right]$$

$$= \frac{1}{2}\left[t\, W_t^2 - \int_0^t (W_s^2 - s)\, ds \right]$$

Find an integration by parts formula for

$$\int_0^t s^2\, W_s^3\, dW_s .$$

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suppose that Y_t satisfies the SDE

$$dY_t = \mu(t, Y_t)\, dt + \sigma(t, Y_t)\, dW_t$$

and $X_t = h(t, Y_t)$ which is once diff in t
& twice " " y

then:

$$dX_t = \partial_y h(t, Y_t)\, dY_t + \partial_t h(t, Y_t)\, dt + \frac{1}{2} \partial_{yy} h(t, Y_t)\, \sigma^2(t, Y_t)\, dt$$

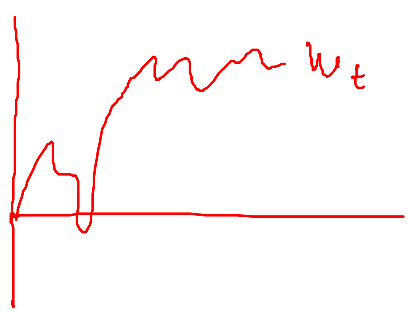
Find an integration by parts formula for $\int_0^t s^2 W_s^3\, dW_s$

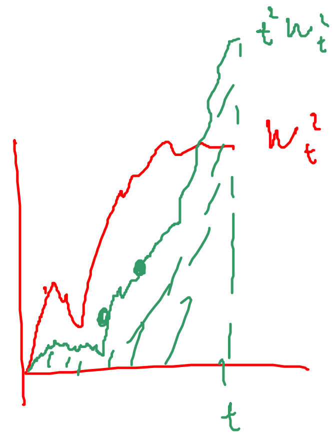
$h(t, y) = t^2 y^4$, $X_t = h(t, W_t)$

$$dX_t = 4\, t^2 W_t^3\, dW_t + \left(2t\, W_t^4 + \frac{1}{2} 4.3.\, t^2 . W_t^2\right) dt$$

$$\int_0^t s^2 W_s^3\, dW_s = \frac{1}{4}\left[\, t^2 W_t^4 - \int_0^t \left(2s W_s^4 + 6 s^2 W_s^2\right) ds \right]$$

$$\int_0^t s^2 W_s^2\, ds$$





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Geometeric Browian motion (GBM)

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t \qquad \text{(Black-Scholes model)}$$

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t$$

$$X_t = \ln S_t \qquad h(y) = \ln y$$

$$dX_t = \left(\frac{1}{S_t}\right) dS_t + 0\, dt + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)\cdot(\sigma S_t)^2\, dW_t$$

$\hookrightarrow \partial_y h(S_t)$ $\qquad \hookrightarrow \underline{\partial_t h(S_t) = 0}$ $\qquad \hookrightarrow \partial_{yy} h(S_t)$

$$\partial_t h(S_t) \neq h'(S_t)\, \partial_t S_t$$

$$h: \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$$
$$(t, y) \mapsto h(t, y)$$

$$\partial_t h(t, S_t) = \partial_t h(t, y)\Big|_{y = S_t}$$

$$= \left(\frac{1}{S_t}\right)(\mu S_t\, dt + \sigma S_t\, dW_t) - \frac{1}{2}\sigma^2\, dW_t$$

$$\Rightarrow dX_t = (\mu - \tfrac{1}{2}\sigma^2)\, dt + \sigma\, dW_t$$

$$X_t - X_0 = (\mu - \tfrac{1}{2}\sigma^2)\, t + \sigma W_t$$

$$\boxed{S_t = S_0\, e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \quad \text{solves} \quad dS_t = \mu S_t\, dt + \sigma S_t\, dW_t}$$

$$S_T = S_0\, e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T}$$

$$\overset{d}{=} S_0\, e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z}$$

$$Z \sim N(0,1)$$

$$\mathbb{E}[S_T] = S_0 \, e^{(\mu - \frac{1}{2}\sigma^2)T} \, \mathbb{E}[e^{\sigma\sqrt{T} Z}]$$

$$= S_0 \, e^{(\mu - \frac{1}{2}\sigma^2)T} \, e^{\frac{1}{2}\sigma^2 T}$$

$$= S_0 \, e^{\mu T}$$

---

given: $S_t = S_0 \, e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$ find the SDE for S_t

$$S_t = h(t, W_t), \qquad h(t,y) = S_0 \, e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma y}$$

$$dS_t = \partial_y h(t, W_t) \cdot dW_t + \partial_t h(t, W_t)\, dt + \frac{1}{2} \partial_{yy} h(t, W_t)\, dt$$

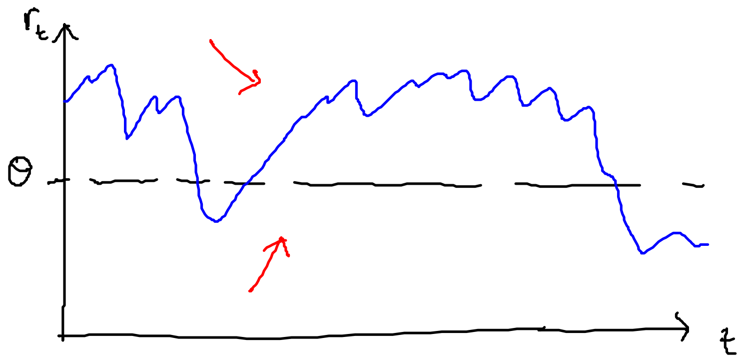
$$= \sigma S_t \, dW_t + (\mu - \tfrac{1}{2}\sigma^2) S_t \, dt + \tfrac{1}{2}\sigma^2 S_t \, dt$$

$$= \mu S_t \, dt + \sigma S_t \, dW_t$$

$\kappa, \theta, \sigma > 0$.

Vasicek Model :

$$r_n - r_{n-1} = \kappa(\theta - r_{n-1})\Delta t + \sigma\sqrt{\Delta t}\, x_n$$



mean-reverting process

(Ornstein - Uhlenbeck)

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\, dW_t$$

$$\left[\begin{array}{ll} \sigma = 0 : & dr_t = -\kappa r_t\, dt \\ \theta = 0 : & \\ & r_t = r_0 e^{-\kappa t} \end{array}\right]$$

$\rightarrow$. try $r_t = e^{-\kappa t} g_t = h(t, g_t)$, $h(t,y) = e^{-\kappa t} y$

$$g_t = e^{\kappa t} r_t = h(t, r_t) , \quad h(t,y) = e^{\kappa t} y$$

$$dg_t = \partial_y h(t, r_t)\, dr_t + \partial_t h(t, r_t)\, dt$$
$$+ \tfrac{1}{2} \partial_{yy} h(t, r_t)\, \sigma^2\, dt$$

$$= e^{\kappa t}\left(\kappa(\theta - r_t)\,dt + \sigma\, dW_t \right)$$
$$+ \kappa e^{\kappa t} r_t\, dt$$
$$+ \tfrac{1}{2}\, 0 \,.\, dt$$

$$\Rightarrow \quad dg_t = \kappa\theta e^{\kappa t}\, dt + e^{\kappa t}\sigma\, dW_t$$

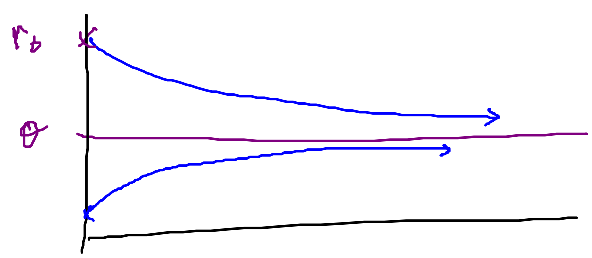
$$g_t - g_0 = \kappa\theta \int_0^t e^{\kappa s}\, ds + \sigma \int_0^t e^{\kappa s}\, dW_s$$

$$\Rightarrow \quad e^{\kappa t} r_t - r_0 = \kappa\theta \frac{e^{\kappa t} - 1}{\kappa} + \sigma \int_0^t e^{\kappa s}\, dW_s$$

$$\Rightarrow$$

$$\boxed{r_t = \underline{r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t})} + \sigma \underline{\int_0^t e^{-\kappa(t-s)}\, dW_s}}$$

?



$$\text{now} \quad \int_0^t e^{-\kappa(t-s)}\, dW_s \sim N(m, v)$$

$$\mathbb{E}\left[\int_0^t e^{-\kappa(t-s)}\, dW_s\right] = 0$$

$$\mathbb{V}\left[\int_0^t e^{-\kappa(t-s)}\, dW_s\right]$$

$$= \mathbb{E}\left[\left(\int_0^t e^{-\kappa(t-s)}\, dW_s\right)^2\right]$$

$$= \mathbb{E}\left[\int_0^t e^{-2\kappa(t-s)}\, ds\right]$$

$$= \int_0^t e^{-2\kappa(t-s)}\, ds$$

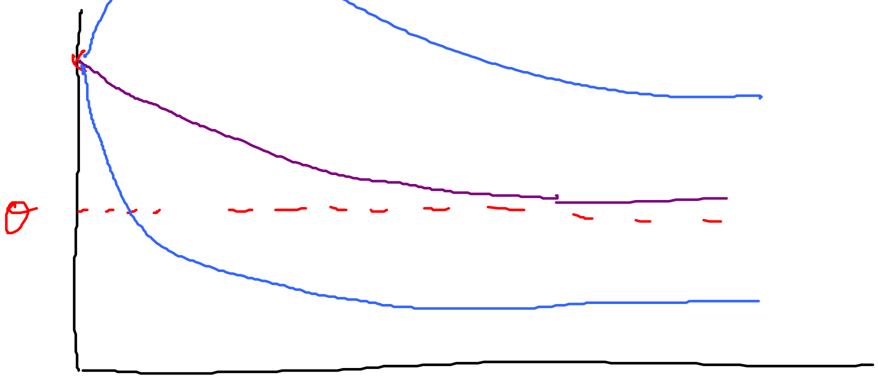
$$= \underline{1 - e^{-2\kappa t}}$$

2κ

$t \ll 1$ $\quad \sigma^2 t$

$$r_t \sim \mathcal{N}\left(r_0 e^{-\kappa t} + \theta(1-e^{-\kappa t})\,;\ \sigma^2\left(\frac{1-e^{-2\kappa t}}{2\kappa}\right)\right)$$

$t \uparrow +\infty \longrightarrow \dfrac{\sigma^2}{2\kappa}$



$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\, dW_t$$

$$r_{t_n} - r_{t_{n-1}} \overset{d}{=} \kappa(\theta - r_{t_{n-1}})\,\Delta t + \sigma\sqrt{\Delta t}\, Z$$

$$\kappa\theta\Delta t + (1-\kappa\Delta t)\, r_{t_{n-1}} + \sigma\sqrt{\Delta t}\, Z$$

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Ito's isommetry.

$$\mathbb{E}\left[\left(\int_0^t g(s, W_s)\, dW_s\right)^2\right]$$

$$= \mathbb{E}\left[\int_0^t (g(s, W_s))^2\, ds\right]$$

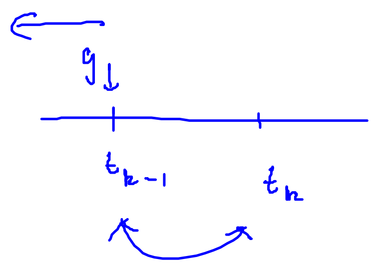
Suppose W_t & B_t are corr. B. motion corr $= \rho$.

$$\mathbb{E}\left[\left(\int_0^t g(s, W_s, B_s)\, dW_s\right)\left(\int_0^t h(s, W_s, B_s)\, dB_s\right)\right]$$

$$= \mathbb{E}\left[\int_0^t g(s, W_s, B_s)\, h(s, W_s, B_s)\, \rho\, ds\right]$$

$$\mathbb{E}\left[\int_0^t g(s, W_s, B_s)\, dW_s\right] = 0$$

take a partition Π



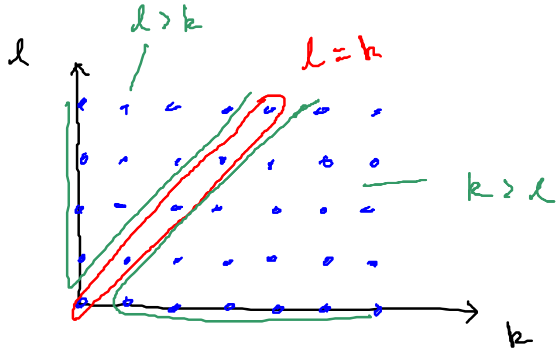
$$\mathbb{E}\left[\sum_k g_{t_{k-1}}\, \Delta W_k\right]$$

$$= \sum_k \mathbb{E}\left[g_{t_{k-1}}\, \Delta W_k\right]$$

$$= \sum_k \mathbb{E}[g_{t_{k-1}}]\, \mathbb{E}[\Delta W_k] = 0$$

For Ito's isommetry take a partition Π for both integrals:

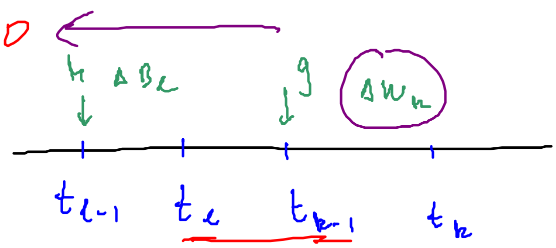
$$I = \mathbb{E}\left[\sum_k g_{t_{k-1}} \underline{\underline{\Delta W_k}} \sum_l h_{t_{l-1}} \underline{\underline{\Delta B_l}} \right]$$



$$\sum_k \mathbb{E}\left[g_{t_{k-1}} h_{t_{k-1}} \Delta W_k \Delta B_k \right]$$

$$+ \sum_{k>l} \mathbb{E}\left[g_{t_{k-1}} h_{t_{l-1}} \Delta W_k \Delta B_l \right] + (l \leftrightarrow k)$$

↳ 0 ↳ 0



↳ $\mathbb{E}\left[g_{t_{k-1}} h_{t_{l-1}} \Delta B_l \right] \underbrace{\mathbb{E}\left[\Delta W_k \right]}_{\to 0}$

$$\mathbb{E}\left[g_{t_{k-1}} h_{t_{k-1}} \Delta W_{t_k} \Delta B_{t_k} \right]$$

$$= \mathbb{E}\left[g_{t_{k-1}} h_{t_{k-1}} \right] \underbrace{\mathbb{E}\left[\Delta W_{t_k} \Delta B_{t_k} \right]}_{\to \rho \Delta t_k}$$

$$\therefore \quad I^{\Pi} = \sum_k \mathbb{E}\left[g_{t_{k-1}} h_{t_{k-1}} \right] \rho \Delta t_k$$

$$= \mathbb{E}\left[\sum_k g_{t_{k-1}} h_{t_{k-1}} \rho \Delta t_k \right]$$

$$\lim_{\|\Pi\| \downarrow 0} I^{\Pi} = \mathbb{E}\left[\lim_{\|\Pi\| \downarrow 0} \sum_k g_{t_{k-1}} h_{t_{k-1}} \rho \, \Delta t_k \right]$$

$$= \mathbb{E}\left[\int_0^t g(s, W_s, B_s) \, h(s, W_s, B_s) \, \rho \, ds \right]$$