Brownian Motion.

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$$r \begin{cases} r + \sigma\sqrt{\Delta t} \\ r - \sigma\sqrt{\Delta t} \end{cases}$$

$$r_n = r_{n-1} + \sigma\sqrt{\Delta t}\, x_n$$

$$= r_0 + \sigma X_n$$

$$0 \begin{cases} \sqrt{\Delta t} \\ -\sqrt{\Delta t} \end{cases}$$

$$X_n = X_{n-1} + \sqrt{\Delta t}\, x_n$$

$$S \begin{cases} S\, e^{\sigma\sqrt{\Delta t}} \\ S\, e^{-\sigma\sqrt{\Delta t}} \end{cases}$$

$$S_n = S_0\, e^{\sigma X_n}$$

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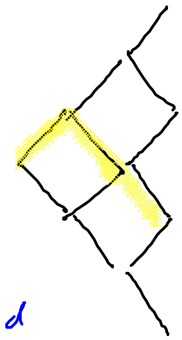
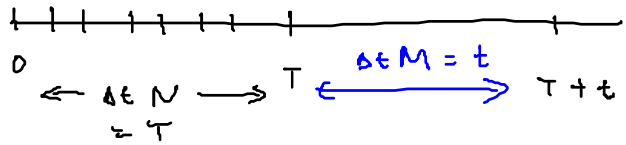
$$\underline{\underline{\sigma = 1}}$$



$$X_n = X_{n-1} + \sigma \sqrt{\Delta t}\, x_n$$

$$\mathbb{P}(x_n = \pm 1) = \frac{1}{2} \qquad x_1, x_2, \ldots \text{ iid}$$



$$\Delta t \downarrow 0 \qquad X_T \xrightarrow[\Delta t \downarrow 0]{d} N(0, \overset{1}{\cancel{\sigma^2}}\, T)$$

by CLT

$$\mathbb{E}^{\mathbb{P}}[X_T] = 0$$

$$\mathbb{V}^{\mathbb{P}}[X_T] = \overset{1}{\cancel{\sigma^2}}\, \Delta t\, N \cdot \mathbb{V}[x_1] = \overset{1}{\cancel{\sigma^2}}\, \Delta t \cdot N \cdot 1 = \overset{1}{\cancel{\sigma^2}}\, T$$

$$\underbrace{X_{T+t} - X_T}_{\text{Increment of } X} \xrightarrow[\Delta t \downarrow 0]{d} N(0, \overset{1}{\cancel{\sigma^2}}\, t)$$

$$\Delta X = X_{T+t} - X_T = \overset{1}{\cancel{\sigma}}\sqrt{\Delta t} \sum_{n=N+1}^{N+M} x_n$$

$$\mathbb{E}^{\mathbb{P}}[\Delta X] = 0$$

$$\mathbb{V}^{\mathbb{P}}[\Delta X] = \overset{1}{\cancel{\sigma^2}}\, \Delta t\, M\, \mathbb{V}^{\mathbb{P}}[x_{N+1}]$$

$$= \overset{1}{\cancel{\sigma^2}}(\Delta t\, M) = \overset{1}{\cancel{\sigma^2}}\, t$$

$$\Delta X_{12} = X_{T_2} - X_{T_1} \qquad \Delta X_{34}$$

$$T_1 < T_2 \le T_3 < T_4$$

$$\Delta X_{12} = \cancel{\sigma}^{1} \sqrt{\Delta t} \sum_{n=N_1+1}^{N_2} x_n$$

$$\Delta X_{34} = \cancel{\sigma}^{1} \sqrt{\Delta t} \sum_{n=N_3+1}^{N_4} x_n$$

are independent!

X_t :

* $X_0 = 0 \quad$ a.s.

* $X_t \sim N(0, t)$

* X_t has independent increments, i.e.

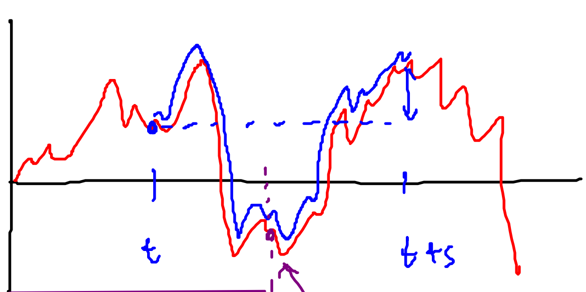
$$X_{T_2} - X_{T_1} \perp X_{T_4} - X_{T_3}$$

$$\forall \; T_1 < T_2 \le T_3 < T_4$$

* X_t has stationary increments, i.e.

$$X_{t+s} - X_t \overset{d}{=} X_s$$

$$\text{lhs} \sim N(0, \underbrace{(t+s) - t}_{=s}) \sim \text{rhs}$$



$$x_s \neq X_{t+s} - X_t$$

- X_t has continuous paths

Such a stochastic process is called a Brownian motion.

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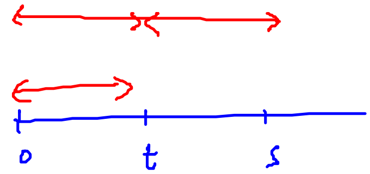
$$X_t \sim N(0,t)$$



$$Y = X_t X_s, \qquad t < s$$

$$\mathbb{E}[Y] = \mathbb{E}[X_t X_s]$$

$$= \mathbb{E}[X_t(X_t + (X_s - X_t))]$$

$$= \mathbb{E}[X_t^2] + \mathbb{E}[\underbrace{X_t \, (X_s - X_t)}_{\text{ind.}}]$$

$$= t + \mathbb{E}[X_t]\mathbb{E}[X_s - X_t]$$

$$= t + 0$$

$$\mathbb{E}[X_t X_s] = \min(t,s)$$

$$\mathbb{E}[X_t^{2n+1}] = 0 \qquad n \in \mathbb{Z}_+$$

b.c. X_t has symmetric dist about 0.

$$\mathbb{E}[X_t^4] = 3t^2$$

$$X_t \overset{d}{=} \sqrt{t}\, Z, \qquad Z \sim N(0,1)$$

$$\Rightarrow \mathbb{E}[X_t^4] = t^2 \, \mathbb{E}[Z^4] = 3t^2$$

$$g(a) = \mathbb{E}[e^{aZ}] = e^{\frac{1}{2}a^2}$$

$$\mathbb{E}[Z^4] = \partial_a^4 g(a) \Big|_{a=0}$$

$$\partial_a g = a g$$

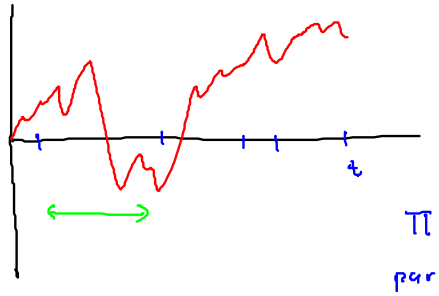
$$\partial_{aa} g = g + a(ag) = (1+a^2) g$$

$$\partial_{aaa} g = 2a\, g + (1+a^2)\, a g$$

$$= (3a + a^3)\, g$$

$$\partial_a^4 g = (3 + 3a^2)\, g + (3a + a^3)\, a g$$

$$\xrightarrow[a \downarrow 0]{} 3$$



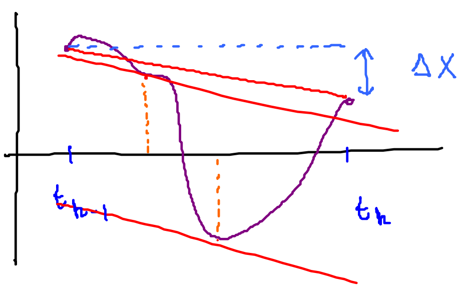
$$\underset{\text{partition}}{\Pi} = \{ 0 = t_0 < t_1 < t_2 < \cdots < t_N = t \}$$

total variation

$$TV \triangleq \lim_{\underbrace{\|\Pi\|}_{L^\infty\text{-norm}} \downarrow 0} \sum_{k=1}^{N} |X_{t_k} - X_{t_{k-1}}|$$

$$\|\Pi\| \triangleq \max_k (t_k - t_{k-1})$$

diff case: X_t



$$\Delta X_k \triangleq X_{t_k} - X_{t_{k-1}}$$

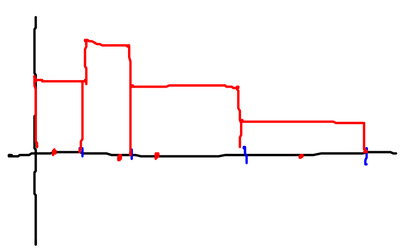
$$\Delta t_k \triangleq t_k - t_{k-1}$$

mean-value thm: $\exists\ t_k^* \in (t_{k-1}, t_k)$ s.t.

$$X'_{t_k^*} = \frac{\Delta X_k}{\Delta t_k} \Rightarrow \Delta X_k = X'_{t_k^*} \Delta t_k$$

$$\Rightarrow TV^\pi = \sum_k |X'_{t_k^*}| \cdot \Delta t_k$$

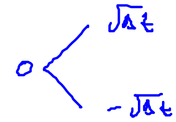
$$\xrightarrow[\|\pi\| \downarrow 0]{} \int_0^t |X'_s|\, ds < +\infty$$



B.mtn: X_t

$$TV^\pi = \sum_k |\Delta X_k|$$

$$\Delta X_k \overset{d}{=} (\Delta t_k)^{1/2} Z_k, \qquad Z_1, Z_2, \ldots \text{ iid} \sim N(0,1)$$



$$TV^\pi \overset{d}{=} \sum_k (\Delta t_k)^{1/2} |Z_k|$$

$$= \sum_k \frac{\Delta t_k}{(\Delta t_k)^{1/2}} |Z_k|$$

$$\ge \underbrace{\frac{1}{\|\pi\|^{1/2}}}_{\xrightarrow[\|\pi\|\downarrow 0]{} +\infty} \underbrace{\sum_k \underbrace{|Z_k|}_{L^1} \Delta t_k}_{< +\infty,\ \to t} \longrightarrow +\infty$$

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here's a little more detail

consider $R^\pi = \sum_k |Z_k| \Delta t_k - t\,\mathbb{E}[|Z|]$ where $Z \sim N(0,1)$

$$\mathbb{E}[R^\pi] = \sum_k \mathbb{E}[|Z_k|]\,\Delta t_k - t\,\mathbb{E}[|Z|]$$

$$= \mathbb{E}[|Z|]\left(\sum_k \Delta t_k - t\right) = 0$$

$$\mathbb{V}[R^\pi] = \sum_k \mathbb{V}[|Z_k|]\,(\Delta t_k)^2$$

$$= \mathbb{V}[|Z|] \sum_k \Delta t_k\, \Delta t_k \le \mathbb{V}[|Z|] \sum_k \Delta t_k\, \|\pi\| \xrightarrow[\|\pi\|\downarrow 0]{} 0$$

$$\Rightarrow \quad \sum_k |Z_k|\,\Delta t_k \xrightarrow[\|\pi\|\downarrow 0]{} t\,\mathbb{E}[|Z|] \quad \text{a.s.}$$

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Quadratic Variation: $[X, X]_t \triangleq \lim\limits_{\|\Pi\| \downarrow 0} \sum\limits_k (\Delta X_k)^2$

<u>Diff Fn</u>: X_t

$$[X, X]_t = \lim_{\|\Pi\|\downarrow 0} \underbrace{\sum_k \left(X'_{t_k^*} \, \Delta t_k \right)^2}_{S}$$

$$S = \sum_k (X'_{t_k^*})^2 \underbrace{\Delta t_k^2}_{\Delta t_k \, \underbrace{\Delta t_k}_{\|\Pi\|}} \le \|\Pi\| \underbrace{\sum_k (X'_{t_k^*})^2 \, \Delta t_k}_{\xrightarrow[\|\Pi\|\downarrow 0]{} \int_0^t (X'_s)^2 \, ds < +\infty}$$

$$\xrightarrow[\|\Pi\|\downarrow 0]{} 0$$

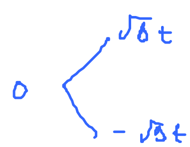
but $[X, X]_t \ge 0$ and $[X, X]_t \le 0$

$$\Rightarrow [X, X]_t = 0$$

<u>B. mtn</u>: X_t

$$[X, X]_t^{\Pi} = \sum_k (\Delta X_k)^2$$

$\hookrightarrow \pm\sqrt{\Delta t_k}$



$$\sum_k \Delta t_k = t$$

$R^{\Pi} = [X, X]_t^{\Pi} - t \xrightarrow[\|\Pi\|\downarrow 0]{} 0$ a.s. via Law large numbers.

$$\mathbb{E}[R^{\Pi}] \xrightarrow[\|\Pi\|\downarrow 0]{} 0, \quad \mathbb{V}[R^{\Pi}] \xrightarrow[\|\Pi\|\downarrow 0]{} 0$$

$$R^{\Pi} = \sum_k (\Delta X_k)^2 - t \qquad = \sum \Delta t_k$$

k

$$= \sum_k \left((\Delta X_k)^2 - \Delta t_k \right)$$

$$\mathbb{E}[\, R^\pi \,] = \sum_k \left(\mathbb{E}[(\Delta X_k)^2] - \Delta t_k \right) = 0$$

since $\Delta X_k \sim \mathcal{N}(0, \Delta t_k)$
$\Rightarrow \mathbb{E}[(\Delta X_k)^2] = \Delta t_k$

$$\mathbb{V}[R^\pi] = \sum_k \mathbb{V}\left((\Delta X_k)^2 - \Delta t_k \right)$$

$$= \sum_k \mathbb{V}[(\Delta X_k)^2]$$

$$\hookrightarrow \mathbb{E}[(\Delta X_k)^4] - \left(\mathbb{E}[(\Delta X_k)^2] \right)^2$$

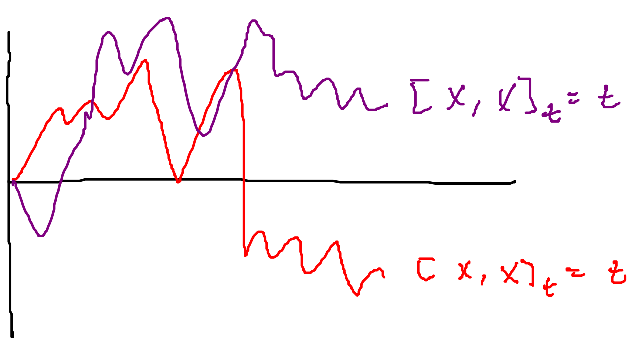
$$= 3\, \Delta t_k^2 - \Delta t_k^2 = 2\, \Delta t_k^2$$

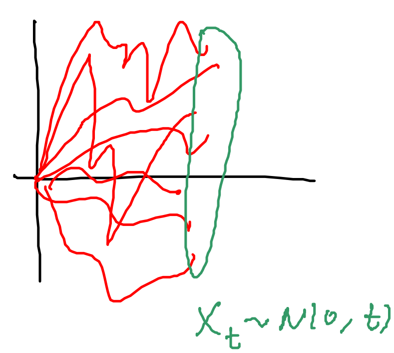
$$= 2 \sum_k \Delta t_k^2$$

$$\leq 2 \underbrace{\left(\sum_k \Delta t_k \right)}_{\to\, t} \| \pi \| \xrightarrow[\|\pi\| \downarrow 0]{} 0$$

$\therefore$ by LLN $\quad R^\pi \xrightarrow[\|\pi\| \downarrow 0]{} 0 \quad$ a.s.

$$\Rightarrow [X, X]_t = t \quad \text{a.s.}$$





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X_t is a B.mtn

$$dY_t = X_t \, dX_t$$

$$Y_t - Y_0 = \int_0^t X_s \, dX_s$$

l.h.s.

$$\hookrightarrow \lim_{\|\pi\| \downarrow 0} \sum_k X_{t_{k-1}} \left(X_{t_k} - X_{t_{k-1}} \right)$$

Ito integral

$$X_t^2 - X_0^2 = \int_0^t d(X_s^2) = 2\int_0^t X_s \, dX_s \quad \text{if } X_t \text{ were diff}$$

seems like $\displaystyle\int_0^t X_s \, dX_s = \frac{1}{2}\left(X_t^2 - t \right)$ a.s.