$$\begin{tikzcd}[column sep=large]
\mathbb{P}\text{- model} \arrow[r, "\text{no-arb}", "\text{dynamic trading}"'] & \text{PDE (deterministic)} \arrow[r, "\text{Feynman}", "\text{Kac}"'] & \mathbb{P}^*\text{- model} \arrow[ll, bend left, "\mathbb{P}^* \sim \mathbb{P} \text{ s.t. relative prices are martingales}"]
\end{tikzcd}$$

Feynman-Kac Theorem:

suppose $h : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ satisfies the PDE

$$\begin{cases} \partial_t h(t,x) + a(t,x)\, \partial_x h(t,x) + \frac{1}{2} b^2(t,x)\, \partial_{xx} h(t,x) = c(t,x)\, h(t,x) \\ h(T,x) = H(x) \end{cases}$$

$\exists$ a probabilistic representation:

$$h(t,x) = \mathbb{E}^{\mathbb{P}^*}_{t,x}\left[e^{-\int_t^T c(u, X_u)\, du} \, H(X_T) \right]$$

and where $X = (X_t)_{t \geq 0}$ satisfies the SDE

$$dX_t = a(t, X_t)\, dt + b(t, X_t)\, dW^*_t$$

and $W^* = (W^*_t)_{t \geq 0}$ is a $\mathbb{P}^*$ - Brownian motion.

recall pricing PDE:

$$\begin{cases} \partial_t g(t,x) \\ + \left(\mu^X(t,x) - \lambda^X(t,x)\, \sigma^X(t,x) \right) \partial_x g(t,x) \end{cases}$$

$$\begin{cases} \quad + \left(\mu^x(t,x) - \lambda^x(t,x)\,\sigma^x(t,x) \right) \partial_x g(t,x) \\ \quad + \frac{1}{2} \left(\sigma^x(t,x)\right)^2 \partial_{xx} g(t,x) \qquad = r(t,x)\, g(t,x) \\ \qquad\qquad g(T,x) = G(x) \end{cases}$$

from Feynman-Kac:

$$g(t,y) = \mathbb{E}^{\mathbb{P}^*}_{t,y}\left[e^{-\int_t^T r(u, Y_u)\,du} \cdot G(Y_T) \right]$$

$$dY_t = \left(\mu^x(t, Y_t) - \lambda^x(t, Y_t)\,\sigma^x(t, Y_t)\right) dt$$

$$\qquad + \sigma^x(t, Y_t)\, dW_t^*$$

$W^* = (W_t^*)_{t \geq 0}$ is a $\mathbb{P}^*$- B.mtn

This could be the end of the story ....

but ... N.B.:

$$dY_t = \mu^x(t, Y_t)\,dt$$

$$\qquad + \sigma^x(t, Y_t) \underbrace{\left(dW_t^* - \lambda^x(t, Y_t)\,dt \right)}$$

$Z = (Z_t)_{t \geq 0}$, $Z_t \hat{=} W_t^* - \int_0^t \lambda^x(u, Y_u)\,du$

(*) ¿ $\exists\ \hat{\mathbb{P}} \sim \mathbb{P}^*$ s.t. Z is a $\hat{\mathbb{P}}$ - B.mtn?

YES! Girsanov's Theorem supplies you with the measure change

$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*}$ a Radon-Nikodym derivative s.t. $(*)$ is true.

Hence, $dY_t = \mu^*(t, Y_t)\,dt + \sigma^*(t, Y_t)\,dZ_t$

Z is a $\hat{\mathbb{P}}$ - B. mtn.

recall that $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$

$$dX_t = \mu^x(t, X_t)\,dt + \sigma^x(t, X_t)\,dW_t$$

$W^* = (W^*_t)_{t\geq 0}$ s.t.

$$W^*_t = W_t + \int_0^t \lambda(u, X_u)\,du$$

$\rightarrow$ W^* is a $\mathbb{P}^*$ - B. mtn where, $(\mathbb{P}^* \sim \mathbb{P})$

$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \dots$ from Girsanov's Theorem.

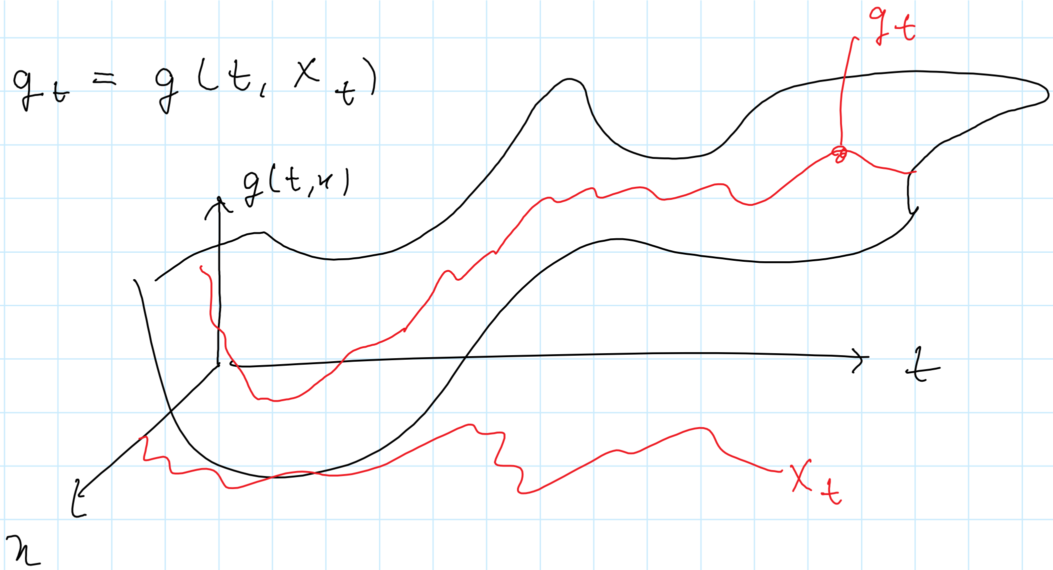
$$\Rightarrow \quad dX_t = \mu^x(t, X_t)\,dt + \sigma^x(t, X_t)\left(dW^*_t - \lambda(t, X_t)\,dt\right)$$

$$= \left(\mu^x(t, X_t) - \lambda(t, X_t)\,\sigma^x(t, X_t)\right)dt$$

$$+ \sigma^x(t, X_t)\,dW^*_t$$

Hence, $g(t, x) = \mathbb{E}^{\mathbb{P}^*}_{t,x}\left[e^{-\int_t^T r(u, X_u)\,du} H(X_T) \right]$

$$g_t = g(t, X_t)$$



$$g_t = \mathbb{E}^{\mathbb{P}^*}_{t,X_t}\left[e^{-\int_t^T r(u,X_u)\,du} \, G(X_T) \right]$$

$$= \mathbb{E}^{\mathbb{P}^*}\Big[\underbrace{e^{-\int_t^T r(u,X_u)\,du}}_{\frac{B_t}{B_T} = \frac{e^{\int_0^t r(u,X_u)\,du}}{e^{\int_0^T r(u,X_u)\,du}}} G(X_T) \,\Big|\, \mathcal{F}_t \Big]$$

$$\frac{g_t}{B_t} = \mathbb{E}^{\mathbb{P}^*}\left[\frac{\overset{g_T}{G(X_T)}}{B_T} \,\middle|\, \mathcal{F}_t \right]$$

$$\Rightarrow \tilde{g}_t = \mathbb{E}^{\mathbb{P}^*}\left[\tilde{g}_T \mid \mathcal{F}_t \right] \qquad \tilde{g} = (\tilde{g}_t)_{t \geq 0}$$

is a $\mathbb{P}^*$ - DOOB martingale $\qquad \tilde{g}_t = g_t / B_t$.

$$\left(\begin{array}{l} \mathbb{E}^{\mathbb{P}^*}[\tilde{g}_s \mid \mathcal{F}_t] \overset{?}{=} \tilde{g}_t \qquad \forall\, s \leq T \\ \quad \| \\ \mathbb{E}^{\mathbb{P}^*}\left[\mathbb{E}^{\mathbb{P}^*}[\tilde{g}_T \mid \mathcal{F}_s] \mid \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{P}^*}[\tilde{g}_T \mid \mathcal{F}_t] = \tilde{g}_t \end{array} \right)$$

We have shown that

$$\boxed{\begin{gathered}\exists \text{ no arb} \iff \exists\ \mathbb{P}^* \sim \mathbb{P} \text{ s.t.} \\ \tilde{g} = (\tilde{g})_{t \geq 0}, \quad \tilde{g}_t = (g_t / B_t) \quad \text{are } \mathbb{P}^*\text{-martingales.}\end{gathered}}$$

Black - Scholes Model

Wednesday, October 12, 2016 3:05 PM

Suppose that $S = (S_t)_{t\geq 0}$ is a price of a traded asset and

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t)$$

GBM geometric Brownian motion

$W = (W_t)_{t\geq 0}$, W - $\mathbb{P}$ Bmtn

hence, $\mu_t^X = S_t\,\mu$, $\sigma_t^X = S_t\,\sigma$

$\mu_t^f = \mu$, $\sigma_t^f = \sigma$

assume $r =$ const.

$$\lambda_t = \frac{\mu_t^f - r_t}{\sigma_t^f} = \frac{\mu - r}{\sigma}$$ is a const here!

Sharpe Ratio
market price of risk.

$$\partial_t g(t,S) + \left(\mu^X(t,S) - \lambda(t,S)\,\sigma^X(t,S)\right)\partial_S g(t,S) + \tfrac{1}{2}\left(\sigma^X(t,S)\right)^2 \partial_{SS}\, g(t,S) = r(t,S)\, g(t,S)$$

$$\Rightarrow \boxed{\begin{cases} \partial_t g(t,S) + r S\, \partial_S g(t,S) + \tfrac{1}{2}\sigma^2 S^2 \partial_{SS}\, g(t,S) = r\, g(t,S) \\ g(T,S) = G(S) \end{cases}}$$

$$g(t,S) = \mathbb{E}_{t,S}^{\mathbb{P}^*}\left[e^{-r(T-t)}\, G(S_T)\right]$$

$$dS_t = (\mu^x(t,S_t) - \lambda(t,S_t)\,\sigma(t,S_t))\,dt + \sigma^x(t,S_t)\,dW_t^*$$

$$= \left(\mu S_t - \frac{\mu - r}{\sigma}\cdot \sigma \cdot S_t\right) dt + \sigma S_t\, dW_t^*$$

$$\Rightarrow \quad dS_t = S_t(r\,dt + \sigma\, dW_t^*)$$

NB: if X_t is traded choose $f_t = X_t$

$$df_t = \mu_t^f f_t\, dt + \sigma_t^f f_t\, dW_t$$

$$dX_t = \underbrace{\mu_t^f X_t}_{\mu_t^x}\, dt + \underbrace{\sigma_t^f X_t}_{\sigma_t^x}\, dW_t$$

$$\lambda_t = \frac{\mu_t^f - r_t}{\sigma_t^f} = \frac{f_t \mu_t^f - r_t f_t}{\sigma_t^f f_t}$$

$$= \frac{\mu_t^x - r_t f_t}{\sigma_t^x}$$

$$\Rightarrow \quad \mu_t^x - \lambda_t \sigma_t^x = r_t f_t = r_t X_t$$

Hence when X is traded ...

$$\begin{cases} \partial_t g(t,x) + r(t,x)\, x\, \partial_x g(t,x) \end{cases}$$

$$\begin{cases} \partial_t g(t,x) + r(t,x)\, x\, \partial_x g(t,x) \\ \quad + \frac{1}{2}\left(\sigma^X(t,x)\right)^2 \partial_{xx} g(t,x) \\ \quad = r(t,x)\, g(t,x) \\ g(T,x) = G(X_T) \end{cases}$$

Suppose 1) $dB_t = r\, B_t\, dt$ traded

2) $dS_t = S_t(\mu\, dt + \sigma\, dW_t)$ traded

3) value a claim paying $G(S_T)$ at T.

suppose $Y \underset{\mathbb{P}}{\sim} \mathcal{N}(0, 1)$

claim $\dfrac{d\mathbb{P}^*}{d\mathbb{P}} = e^{-\frac{1}{2}a^2 - aY}$ is valid R-N
Radon-Nikodym derivative.

1) clearly $\dfrac{d\mathbb{P}^*}{d\mathbb{P}} > 0$ P-a.s

2) $\mathbb{E}^{\mathbb{P}}\left[\dfrac{d\mathbb{P}^*}{d\mathbb{P}}\right] = e^{-\frac{1}{2}a^2}\,\mathbb{E}^{\mathbb{P}}\left[e^{-aY}\right]$

$= e^{-\frac{1}{2}a^2} \cdot e^{\frac{1}{2}a^2}$

$= 1$

¿ What is $\mathbb{P}^*$ - distribution of Y ?

$\mathbb{E}^{\mathbb{P}^*}\left[e^{iuY}\right]$

$= \mathbb{E}^{\mathbb{P}}\left[e^{iuY}\dfrac{d\mathbb{P}^*}{d\mathbb{P}}\right]$

$= \mathbb{E}^{\mathbb{P}}\left[e^{iuY} \cdot e^{-\frac{1}{2}a^2 - aY}\right]$

$= e^{-\frac{1}{2}a^2}\,\mathbb{E}^{\mathbb{P}}\left[e^{(iu-a)Y}\right]$

$= e^{-\frac{1}{2}a^2}\, e^{\frac{1}{2}(iu-a)^2}$ $\rightarrow \frac{1}{2}\left[-u^2 - 2iau + a^2\right]$

$= e^{-iau - \frac{1}{2}u^2}$

$$\Rightarrow \quad Y \underset{\mathbb{P}^*}{\sim} \mathcal{N}(-a, 1)$$

hence $\quad Y^* = Y + a \underset{\mathbb{P}^*}{\sim} \mathcal{N}(0, 1)$

$\mathbb{P}$-B.motion.

now take $\quad W = (W_t)_{t\geq 0}$ ∧ $\quad W_t \underset{\mathbb{P}}{\sim} \mathcal{N}(0, t)$

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{-\frac{1}{2}\int_0^T a_u^2\, du - \int_0^T a_u\, dW_u} \qquad (a_t)_{t\geq 0}\ \mathcal{F}\text{-adapted}$$

1) clearly $\quad \frac{d\mathbb{P}^*}{d\mathbb{P}} > 0 \quad$ a.s.

2) $\quad \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{P}^*}{d\mathbb{P}}\right] = \mathbb{E}^{\mathbb{P}}\left[e^{-\frac{1}{2}\int_0^T a_u^2\, du - \int_0^T a_u\, dW_u}\right]$

$$g_t = \exp\left\{-\frac{1}{2}\int_0^t a_u^2\, du - \int_0^t a_u\, dW_u\right\}$$

$$\Longleftrightarrow \quad dg_t = -g_t\, a_t\, dW_t \quad , \quad g_0 = 1$$

$$f_t = f(g_t) = \log(g_t) \quad ; \quad f(x) = \log x.$$

Ito's lemma

$$df_t = 0 \cdot \partial_x \log(g_t)\, dt$$
$$+ \frac{1}{2}(-g_t\, a_t)^2 \cdot \partial_{xx}(\log(g_t))\, dt$$
$$+ (-g_t\, a_t)\, \partial_x \log(g_t)\, dW_t$$

$$+ (-g_t a_t)\, \partial_x \log(g_t)\, dW_t$$

$$= -\frac{1}{2} a_t^2\, dt - a_t\, dW_t$$

$$\Rightarrow f_t - f_0 = -\frac{1}{2}\int_0^t a_u^2\, du - \int_0^t a_u\, dW_u$$

$$\Rightarrow \log\left(\frac{g_t}{g_0}\right) = -\frac{1}{2}\int_0^t a_u^2\, du - \int_0^t a_u\, dW_u$$

$$\Rightarrow g_t = \underbrace{g_0}_{\hookrightarrow 1}\, e^{-\frac{1}{2}\int_0^t a_u^2\, du - \int_0^t a_u\, dW_u}$$

$$g_t \xrightarrow[t \uparrow T]{} \frac{d\mathbb{P}^*}{d\mathbb{P}} \qquad \left(\text{want } \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{P}^*}{d\mathbb{P}}\right] = 1\right)$$

$$dg_t = -a_t\, g_t\, dW_t$$

$$\underbrace{g_T}_{\frac{d\mathbb{P}^*}{d\mathbb{P}}} - \underbrace{g_0}_{\hookrightarrow 1} = -\int_0^T a_u\, g_u\, dW_u$$

$$\mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{P}^*}{d\mathbb{P}}\right] - 1 = 0$$

¿ want to know what is $\mathbb{P}^*$ - properties of W ?

$$W^* = W + \int^t a \ldots$$

$$W_t^* = W_t + \int_0^t a_u \, du$$

$$\mathbb{E}^{\mathbb{P}^*}\left[e^{iu W_t^*} \right] \qquad , \qquad t \le T$$

$$= \mathbb{E}^{\mathbb{P}}\left[e^{iu W_t^*} \cdot \frac{d\mathbb{P}^*}{d\mathbb{P}} \right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[e^{iu W_t^*} \cdot \underbrace{e^{-\frac{1}{2}\int_0^T a_u^2 \, du - \int_0^T a_u \, dW_u}}_{g_T} \right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[e^{iu W_t^*} \, g_t \cdot (g_T / g_t) \right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[e^{iu W_t^*} g_t \quad (g_T / g_t) \,\middle|\, \mathcal{F}_t \right] \right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[e^{iu W_t^*} \, \mathbb{E}^{\mathbb{P}}\left[g_T \mid \mathcal{F}_t \right] \right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[e^{iu W_t^*} \, g_t \right]$$

$$\int_0^t dW_u$$

$$= \mathbb{E}^{\mathbb{P}}\left[\exp\left\{ iu \left(W_t + \int_0^t a_u \, du \right) - \frac{1}{2}\int_0^t a_u^2 \, du - \int_0^t a_u \, dW_u \right\} \right]$$

$$\alpha$$

$$\{ \quad \} = iu \int_0^t a_u \, du - \frac{1}{2}\int_0^t a_u^2 \, du$$

α

$$-\int_0^t \underbrace{(i\alpha - a_u)}_{b_u}\, dW_u$$

$$b_u^2 = a_u^2 - 2i\alpha a_u - \alpha^2$$

$$= -\frac{1}{2}\int_0^t (b_u^2 + \alpha^2)\, du - \int_0^t b_u\, dW_u$$

$$\mathbb{E}^{\mathbb{P}^*}\left[e^{i\alpha W_t^*} \right] = e^{\frac{1}{2}\alpha^2 t} \underbrace{\mathbb{E}^{\mathbb{P}}\left[e^{-\frac{1}{2}\int_0^t b_u^2\, du - \int_0^t b_u\, dW_u} \right]}_{1}$$

$$\Rightarrow \quad W_t^* \underset{\mathbb{P}^*}{\sim} \mathcal{N}(0, t)$$