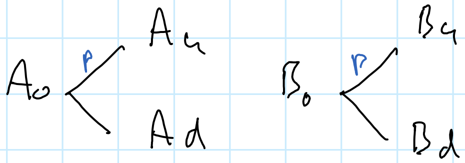
Wednesday, September 21, 2016 2:07 PM

$p \in (0,1)$



risky asset

$B_t > 0$ a.s. $\forall t$

numeraire asset

To avoid arbitrage:

↳ a strategy (α, β) s.t. i) $V_0 = 0$

ii) a. $\mathbb{P}(V_1 \geq 0) = 1$

b. $\mathbb{P}(V_1 > 0) > 0$

$$\frac{A_d}{B_d} \; \underset{>}{<} \; \frac{A_0}{B_0} \; \underset{>}{<} \; \frac{A_u}{B_u}$$

$$\nexists \text{ arbitrage} \iff \exists \mathbb{Q} \sim \mathbb{P} \text{ s.t. } \frac{A_0}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_1}{B_1}\right]$$

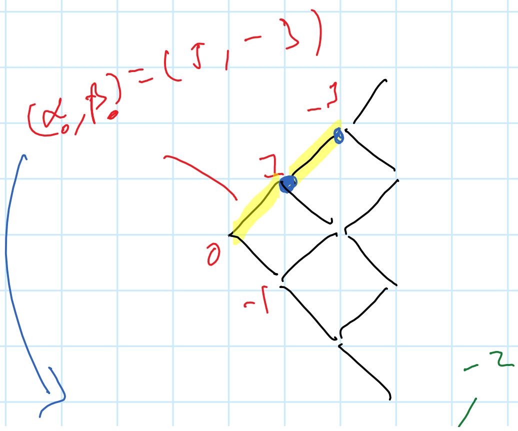
Fundamental Theorem of Finance / Asset Pricing.

multi-step arbitrage strategy is a self-financing strategy $(\alpha_t, \beta_t)_{t \geq 0} \in \mathcal{F}_t$

i) $V_0 = 0$

ii) $\exists\, t > 0$, s.t. a) $\mathbb{P}(V_t \geq 0) = 1$

b) $\mathbb{P}(V_t > 0) > 0$



$V_0 = \alpha_0 A_0 + \beta_0 B_0 = 0$

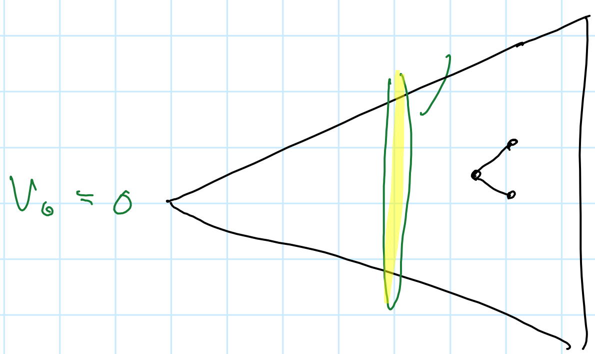
rebalance (α_1, β_1) s.t. $V_1 = \alpha_0 A_1 + \beta_0 B_1$

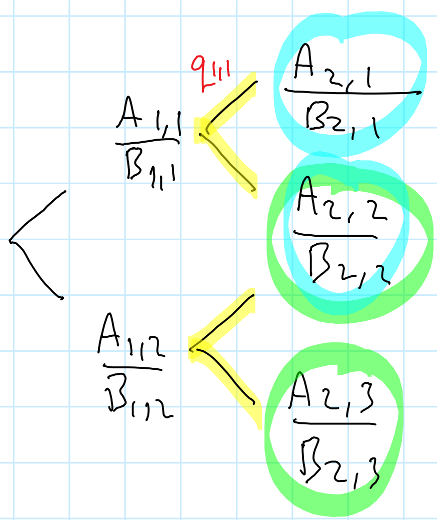
$$= \alpha_1 A_1 + \beta_1 B_1$$

self-financing

$$\boxed{(\alpha_t - \alpha_{t-1}) A_t + (\beta_t - \beta_{t-1}) B_t = 0 \quad \forall t > 0}$$

a) 2D) hold





$A_{t,j}$ — j: state label, t: time step

$$\frac{A_{1,1}}{B_{1,1}} = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_2}{B_2} \,\middle|\, \begin{matrix} A_1 = A_{1,1} \\ B_1 = B_{1,1} \end{matrix} \right]$$

$$\frac{A_{1,2}}{B_{1,2}} = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_2}{B_2} \,\middle|\, \begin{matrix} A_1 = A_{1,2} \\ B_1 = B_{1,2} \end{matrix} \right]$$

$$\frac{A_1}{B_1} = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_2}{B_2} \,\middle|\, \sigma((A_1, B_1)) \right]$$

$\hookrightarrow A_1, B_1$

$$\frac{A_t}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_{t+1}}{B_{t+1}} \,\middle|\, \sigma((A_t, B_t)) \right]$$

$$\frac{A_t}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\frac{A_{t+2}}{B_{t+2}} \,\middle|\, \sigma\left(\left(\begin{matrix} A_t, A_{t+1}, \\ B_t, B_{t+1} \end{matrix} \right) \right) \right] \,\middle|\, \sigma((A_t, B_t)) \right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\frac{A_{t+2}}{B_{t+2}} \,\middle|\, \sigma((A_t, B_t)) \right]$$

$\nexists$ arbitrage $\iff \exists\, \mathbb{Q} \sim \mathbb{P}$ s.t. all traded assets

$$\frac{A_t}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_s}{B_s} \,\middle|\, \mathcal{F}_t \right], \ \forall s \geq t$$

(B is a numeraire asset)

$\mathbb{Q}$ suppose i) $\exists\ \mathbb{Q} \sim \mathbb{P}$ s.t. $\dfrac{A_t^{(m)}}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\dfrac{A_s^{(m)}}{B_s} \,\middle|\, \mathcal{F}_t \right] \ \forall s > t$

$m = 1, \ldots, N$

Thm: If $\mathbb{Q}$ holds then $\nexists$ arbitrage.

<u>Proof:</u> suppose $\exists$ a self-financing strategy

$(\alpha, \beta) = (\alpha_t, \beta_t)_{t \geq 0}$ and a time T

s.t. a) $\mathbb{P}(V_T \geq 0) = 1$

b) $\mathbb{P}(V_T > 0) > 0$

first observe:

a) $\Rightarrow \mathbb{P}\left(\dfrac{V_T}{B_T} \geq 0 \right) = 1 \ \xRightarrow{\text{b/c } \mathbb{Q} \sim \mathbb{P}} \ \mathbb{Q}\left(\dfrac{V_T}{B_T} \geq 0 \right) = 1$

b) $\Rightarrow \mathbb{P}\left(\dfrac{V_T}{B_T} > 0 \right) > 0 \Rightarrow \mathbb{Q}\left(\dfrac{V_T}{B_T} > 0 \right) >$

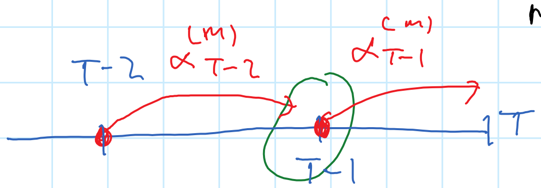
b) $\Rightarrow \mathbb{P}\left(\frac{V_T}{B_T} > 0\right) > 0 \Rightarrow \mathbb{Q}\left(\frac{V_T}{B_T} > 0\right) > 0$

next note that $\mathbb{Q} \Rightarrow \frac{V_t}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{V_T}{B_T} \,\Big|\, \mathcal{F}_t\right]$

b/c V is self-financing

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{V_T}{B_T} \,\Big|\, \mathcal{F}_{T-1}\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{\sum_{m=1}^{N} \alpha_{T-1}^{(m)} A_T^{(m)}}{B_T} \,\Big|\, \mathcal{F}_{T-1}\right]$$

$$= \sum_{m=1}^{N} \alpha_{T-1}^{(m)} \underbrace{\mathbb{E}^{\mathbb{Q}}\left[\frac{A_T^{(m)}}{B_T} \,\Big|\, \mathcal{F}_{T-1}\right]}_{\frac{A_{T-1}^{(m)}}{B_{T-1}} \text{ by } \mathbb{Q}}$$



$$\Rightarrow \mathbb{E}^{\mathbb{Q}}\left[\frac{V_T}{B_T} \,\Big|\, \mathcal{F}_{T-1}\right] = \frac{\sum_{m=1}^{N} \alpha_{T-1}^{(m)} A_{T-1}^{(m)}}{B_{T-1}}$$

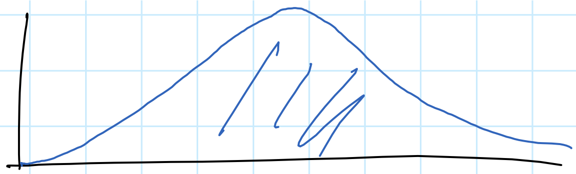
$$= \frac{\sum_{m=1}^{N} \alpha_{T-2}^{(m)} A_{T-1}^{(m)}}{B_{T-1}} = \frac{V_{T-1}}{B_{T-1}}$$

---

recall we have $\mathbb{Q}(\tilde{V}_T \geq 0) = 1$

$\mathbb{Q}(\tilde{V}_T > 0) > 0$

$$\Rightarrow \tilde{V}_0 = \mathbb{E}^{\mathbb{Q}}[\tilde{V}_T] > 0$$



$\Rightarrow \quad V_0 > 0 \quad$ (b/c $B_0 > 0$)

$\therefore \nexists$ no arbitrage $\qquad$ Q.E.D.

CRR - model

$$T = N \Delta t$$

$$A_{t_n} = A_{t_{n-1}} e^{\sigma \sqrt{\Delta t}\, x_n}, \quad t_n = n \Delta t, \; n = 1 \cdots N$$

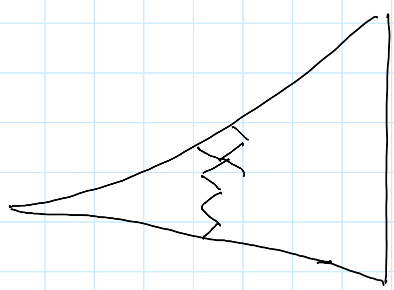
$x_1, x_2, \ldots$ iid Bernoulli ± 1

$$\mathbb{P}(x_1 = +1) = \frac{1}{2}\left(1 + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \sqrt{\Delta t}\right)$$

$$B_{t_n} = B_{t_{n-1}} e^{r \Delta t}$$

$$\lim_{\Delta t \downarrow 0} \log\left(\frac{A_T}{A_0}\right) \underset{\mathbb{P}}{\sim} \mathcal{N}\left((\mu - \tfrac{1}{2}\sigma^2) T \, ; \; \sigma^2 T \right)$$

$$\underset{\mathbb{Q}}{\sim} \mathcal{N}\left((r - \tfrac{1}{2}\sigma^2) T \, ; \; \sigma^2 T \right)$$



$$q = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

$$= \frac{1}{2}\left(1 + \frac{r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + \cdots$$

Value an option on the asset.

call option

option to buy an asset for a price → strike (K) fixed amount at a ...

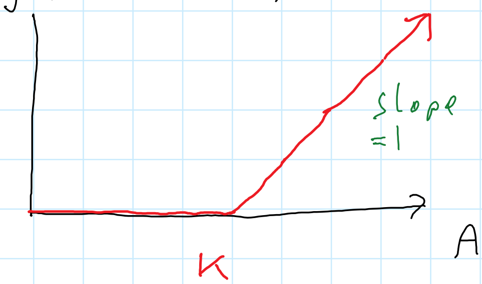
put option

option to sell an asset for a price fixed ... ...

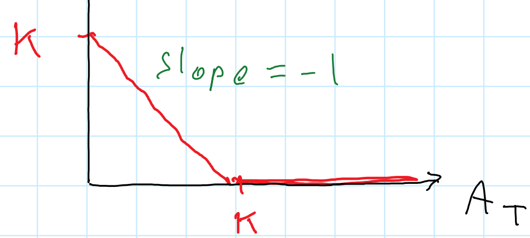
asset for a price → strike (K)
fixed now at a
agreed date. ↳ maturity (T)

asset for a price
fixed now at a
agreed date

call payoff $= \varphi(A_T)$



put payoff



$$\varphi^{call}(A_T) = \max(A_T - K, 0) = (A_T - K)_+$$

$$\varphi^{put}(A_T) = \max(K - A_T, 0) = (K - A_T)_+$$

$(g_t)_{t\geq 0}$ price process for a call option.

$\exists\ \mathbb{Q} \sim \mathbb{P}$ s.t. $\quad \dfrac{g_0}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\dfrac{g_T}{B_T} \,\Big|\, \mathcal{F}_0 \right]$

$(\mathbb{E}_t[\cdot] \triangleq \mathbb{E}[\cdot \mid \mathcal{F}_t])$

$$\Rightarrow \frac{g_0}{B_0} = \mathbb{E}_0^{\mathbb{Q}}\left[\frac{(A_T - K)_+}{B_T} \right]$$

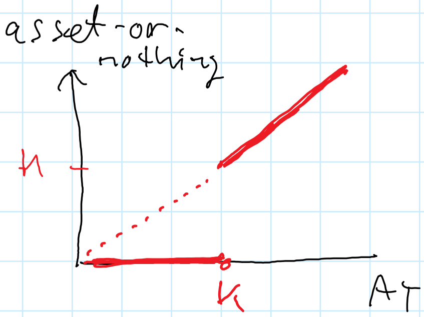
$$(A_T - K)_+ = (A_T - K)\, \mathbb{1}_{A_T > K}$$

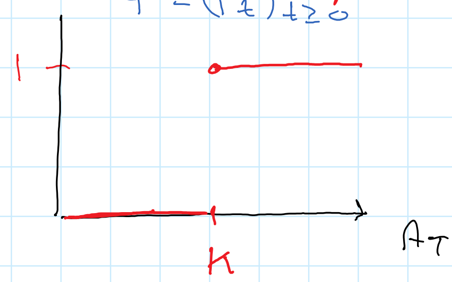
$\left[\mathbb{1}_{\mathcal{C}}(\omega) = \begin{cases} 1 & \text{if } \omega \in \mathcal{C} \\ 0 & \text{otherwise} \end{cases} \quad \text{i.e. } \begin{matrix} 1 \text{ if } \mathcal{C} \text{ occurs} \\ 0 \text{ otherwise} \end{matrix} \right]$

$$\Rightarrow (A_T - K)_+ = \underbrace{A_T \mathbb{1}_{A_T > K}}_{\substack{\text{asset-or-nothing} \\ f = (f_t)_{t\geq 0}}} - K \underbrace{\mathbb{1}_{A_T > K}}_{\substack{\text{digital call} \\ d = (d_t)_{t \geq 0}}}$$

asset-or-nothing

asset-or-nothing $f = (f_t)_{t \ge 0}$ digital call $\ell = (\ell_t)_{t\ge 0}$





Pricing operator is linear, so value each linearly ..

$$\frac{\ell_0}{B_0} = \mathbb{E}_0^{\mathbb{Q}^B}\left[\frac{K\,\mathbb{1}_{A_T > K}}{B_T}\right]$$

$$= \frac{K}{B_T}\,\mathbb{Q}_0^B(A_T > K)$$

if (B_t) is deterministic (not a stochastic process)

$$\frac{f_0}{B_0} = \mathbb{E}_0^{\mathbb{Q}^B}\left[\frac{A_T\,\mathbb{1}_{A_T > K}}{B_T}\right]$$

$$\frac{f_0}{A_0} = \mathbb{E}_0^{\mathbb{Q}^A}\left[\frac{A_T\,\mathbb{1}_{A_T > K}}{A_T}\right]$$

if $A_t > 0$ a.s. $\forall t$

$$= \mathbb{Q}_0^A(A_T > K)$$

$$g_0 = f_0 - \ell_0$$

$$= A_0\,\mathbb{Q}^A(A_T > K) - \frac{B_0}{B_T}\,K\,\mathbb{Q}^B(A_T > K)$$

if r & A are uncorrelated $P_0(T)$

$e^{-\int_0^T r_s\,ds}$ deterministic

$$\mathbb{Q}^B(A_T > K)$$

$$= \mathbb{Q}^B\left(\log\left(\frac{A_T}{A_0}\right) > \underbrace{\log\left(\frac{K}{A_0}\right)}_{\psi}\right)$$

in CRR as $\Delta t \downarrow 0$ $\quad \log\left(\frac{A_T}{A_0}\right) \overset{d}{=} \underbrace{\left(r - \tfrac{1}{2}\sigma^2\right)T}_{\gamma} + \underbrace{\sigma\sqrt{T}}_{\eta}\,Z$

in CRR as $\Delta t \downarrow 0$ $\quad \log\left(\frac{A_T}{A_0}\right) \overset{d}{=} \left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\, Z$

$Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$

$$\mathbb{Q}^B(A_T > K) = \mathbb{Q}^B(\gamma + \eta Z > \psi)$$

$$= \mathbb{Q}^B\left(Z > \frac{\psi - \gamma}{\eta}\right)$$

$$= \mathbb{Q}^B\left(Z < \frac{\gamma - \psi}{\eta}\right)$$

$$= \Phi\left(\frac{\gamma - \psi}{\eta}\right)$$

$$\boxed{\mathbb{Q}^B(A_T > K) = \Phi\Bigg(\underbrace{\frac{\log(A_0/K) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}}_{d_-}\Bigg)}$$

$$\frac{B_{t_{n-1}}}{A_{t_{n-1}}} = \mathbb{E}^{\mathbb{Q}^A}\left[\frac{B_{t_n}}{A_{t_n}} \,\Big|\, \mathcal{F}_{t_{n-1}}\right]$$

$$\left(\frac{B_{t_{n-1}}}{A_{t_{n-1}}}\right) \xrightarrow{q^A} \left(\frac{B_{t_{n-1}}}{A_{t_{n-1}}}\right)\frac{e^{r\Delta t}}{e^{\sigma\sqrt{\Delta t}}}$$

$$\left(\frac{B_{t_{n-1}}}{A_{t_{n-1}}}\right) \longrightarrow \left(\frac{B_{t_{n-1}}}{A_{t_{n-1}}}\right)\frac{e^{r\Delta t}}{e^{-\sigma\sqrt{\Delta t}}}$$

$$\Rightarrow \quad 1 = \frac{e^{r\Delta t}}{e^{\sigma\sqrt{\Delta t}}}\, q^A + \frac{e^{r\Delta t}}{e^{-\sigma\sqrt{\Delta t}}}\,(1 - q^A)$$

$$\Rightarrow \quad e^{-r\Delta t} = e^{-\sigma\sqrt{\Delta t}}\, q^A + e^{\sigma\sqrt{\Delta t}}\,(1-q^A)$$

$$\Rightarrow \quad q^A = \frac{e^{-r\Delta t} - e^{\sigma\sqrt{\Delta t}}}{e^{-\sigma\sqrt{\Delta t}} - e^{\sigma\sqrt{\Delta t}}}$$

$$= \frac{e^{\sigma\sqrt{\Delta t}} - e^{-r\Delta t}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

$$= \frac{(\cancel{1} + \sigma\sqrt{\Delta t} + \tfrac{1}{2}\sigma^2\Delta t) - (\cancel{1} - r\Delta t) + \cdots}{(\cancel{1} + \sigma\sqrt{\Delta t} + \cancel{\tfrac{1}{2}\sigma^2}\Delta t) - (\cancel{1} - \sigma\sqrt{\Delta t} + \cancel{\tfrac{1}{2}\sigma^2}\Delta t) + \cdots}$$

$$= \frac{\sigma\sqrt{\Delta t} + (r + \tfrac{1}{2}\sigma^2)\Delta t + \cdots}{2\sigma\sqrt{\Delta t} + \cdots}$$

$$q^A = \frac{1}{2}\left(1 + \frac{r + \tfrac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + \cdots$$

$$q = \frac{1}{2}\left(1 + \frac{r - \tfrac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + \cdots$$

$$p = \frac{1}{2}\left(1 + \frac{\mu - \tfrac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right)$$

Hence, as $\Delta t \downarrow 0$, $\log\left(\frac{A_T}{A_0}\right) \overset{d}{=} (r + \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z^A$

$$Z^A \underset{\mathbb{Q}^A}{\sim} \mathcal{N}(0,1)$$

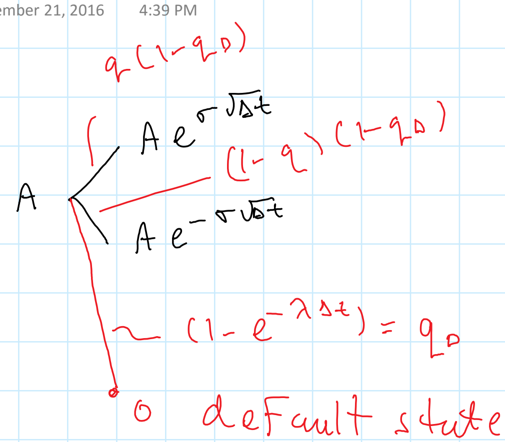
and $\mathbb{Q}^A(A_T > K) = \Phi\left(\overbrace{\frac{\log(A_0/K) + (r + \tfrac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}^{d_+}\right)$

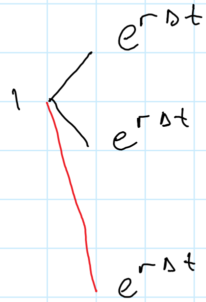
put all together .. in CRR

$$\boxed{g_0 = A_0\, \Phi(d_+) - K e^{-rT}\, \Phi(d_-)}$$

↳ Black-Scholes Formula

$$d_\pm = \frac{\log(A_0/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$





when assets have more outcomes than there are traded assets, markets are said to be incomplete.

$\mathbb{Q}$ is then <u>not</u> unique

prices, in general, satisfy no arbitrage range of prices

$$A_T = \tilde{A}_T \, \mathbb{1}_{\tau > T}$$

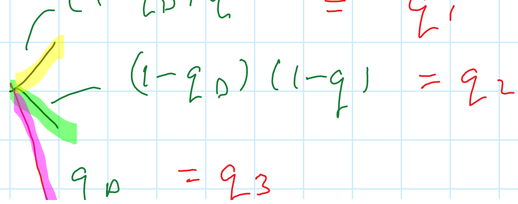
default free asset dynamics

first arrival of a Poisson process intensity λ.

$$(K - A_T)_+ = (K - \tilde{A}_T \, \mathbb{1}_{\tau > T})_+$$

$$= \ldots$$

MC:



$q_d = q_3$

1. generate $u \rightsquigarrow U(0,1)$

2. 

1 0 0

0 0 0

1 1 0

$CQ = \text{cumsum}(q)$

$\text{sum}(u > CQ) + 1$

$$\begin{bmatrix} q_1 \\ q_1+q_2 \\ q_1+q_2+q_3 \end{bmatrix}$$

3. repeat

$T=1, \quad \Delta T = ? \quad$ until converge $1 \triangleright p.$

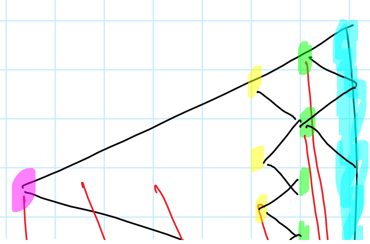
$$A_T^{(1)}, \ \ldots\ , \ A_T^{(M)} \qquad \text{asset terminal value}$$

$$\mathbb{Q}^{(1)} = (K - A_T^{(1)})_+ \quad \ldots \quad (K - A_T^{(M)})_+ \qquad \text{payoff}$$

$\hookrightarrow \mathbb{Q}^{(M)}$

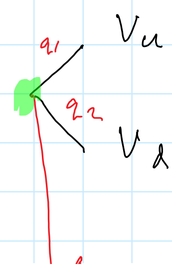
$$V = e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\left[(K - A_T)_+ \right]$$

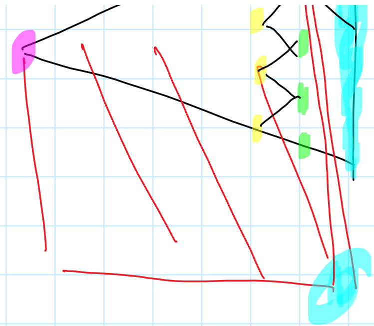
$$\sim e^{-rT}\left[\underbrace{\frac{1}{M}\sum_{m=1}^{M} \mathbb{Q}^{(m)}}_{\text{sample mean}} \pm \frac{1}{\sqrt{M}} \overbrace{\sqrt{\frac{1}{M}\sum_{m=1}^{M}\left(\mathbb{Q}^{(m)} - \bar{\mathbb{Q}}\right)^2}}^{\sqrt{\text{sample variance}}} \right]$$



$(K - A_T)_+$

$$\frac{V_0}{B_0} = q_1 \frac{V_u}{B_u} + q_2 \frac{V_d}{B_d}$$





$$\frac{V_0}{B_0} = q_1 \frac{V_u}{B_u} + q_2 \frac{V_d}{B_d} + q_3 \frac{K}{B_{def}}$$

q_3

K

$\Rightarrow$

$$V_0 = e^{-r\Delta T}\left(q_1 V_u + q_2 V_d + q_3 K\right)$$

American versus European option

exercise on any date up to and including mat. date	maturity date is the only date on which you can exercise

American Put

$$A_0 \overset{q}{\left\langle\begin{matrix} A_u \\ A_d \end{matrix}\right.} \qquad B_0 \overset{q}{\left\langle\begin{matrix} B_0 e^{r\Delta t} \\ B_0 e^{r\Delta t} \end{matrix}\right.}$$

(hold) continuation value

$$V_0^C = e^{-r\Delta T}\left(q V_u + (1-q) V_d\right) \overset{q}{\left\langle\begin{matrix} (K - A_u)_+ = V_u \\ (K - A_d)_+ = V_d \end{matrix}\right.}$$

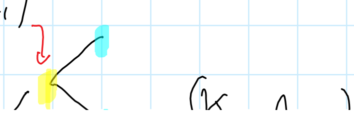
$$V_0^I = (K - A_0)_+$$

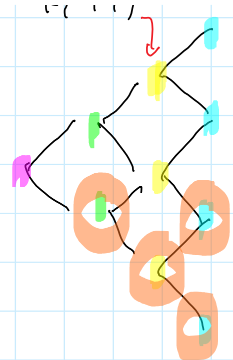
intrinsic value or exercise value.

you have flexibility to choose

$$\max\left(V_0^I, V_0^C\right) = V_0$$

$$\max\left(V_{T-1}^I, V_{T-1}^C\right)$$

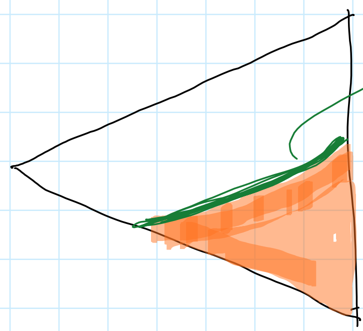




$(K - A_T)_+$

= optimal to exercise

$$\boxed{V(t, S) = \sup_{\tau} \mathbb{E}^{\mathbb{Q}}_t \left[e^{-r\tau} (K - A_\tau)_+ \right]}$$



optimal exercise boundary

protected put payoff $\quad (K - A_{\tau^e})_+ \, \mathbb{1}_{\tau_d > \tau^e}$

exercise time