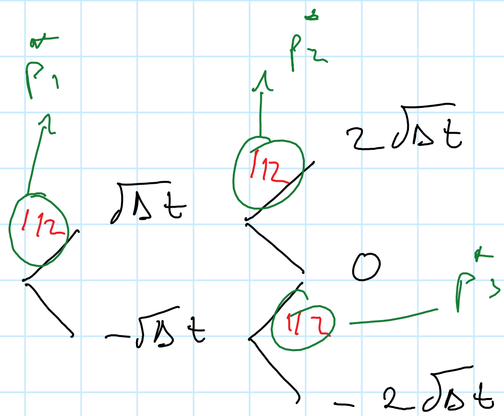
$W_0 = 0$



$\ldots$

$p_k^* \in (0,1)$

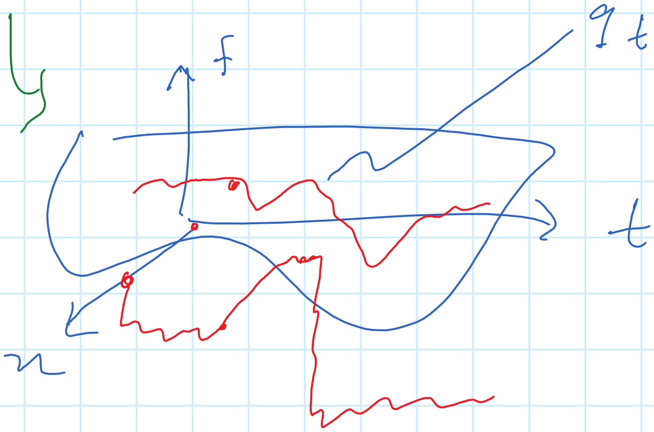
$$W_{n\Delta t} = W_{(n-1)\Delta t} + \sqrt{\Delta t}\; x_n$$

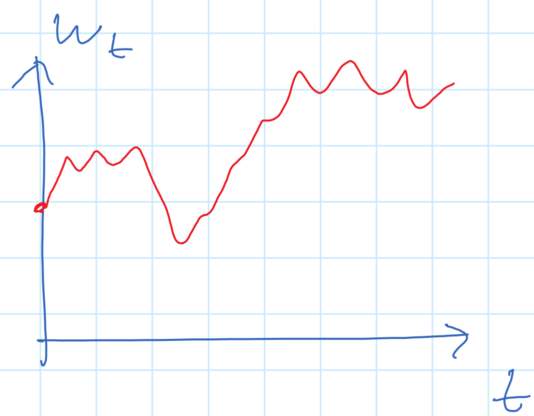
$\hookrightarrow \pm 1$ Bernoulli

$p = 1/2$.

$F(t, x)$, $F(t, W_t) = g_t$

$f : \mathbb{R}_+ \times \mathbb{R} \longmapsto \mathbb{R}$





$$\begin{cases} \partial_t f + a(t,x)\,\partial_x f + \frac{1}{2}\,\sigma^2(t,x)\,\partial_{xx} f = c(t,x)\, f \\ \qquad\qquad\qquad\qquad f(T,x) = F(x) \end{cases}$$

$X_T^{t,x}$

$$f(t,x) = \mathbb{E}^*_{t,x}\left[F(X_T)\, e^{-\int_t^T c(u, X_u)\, du} \right]$$

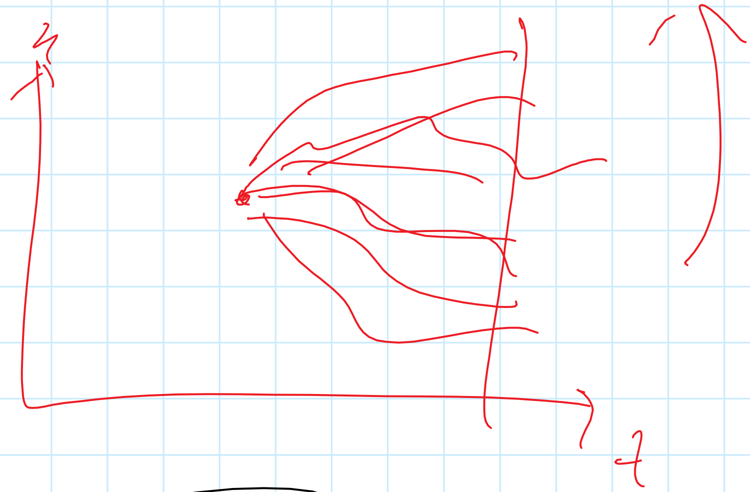
$$F(t,x) = \mathbb{E}_{t,x}\left[F(X_T)\, e^{-\int_t^T r(u,X_u)\,du} \right]$$

$$\to \quad dX_t = a(t, X_t)\,dt + b(t, X_t)\,dW_t^*$$

$t \hookrightarrow s$

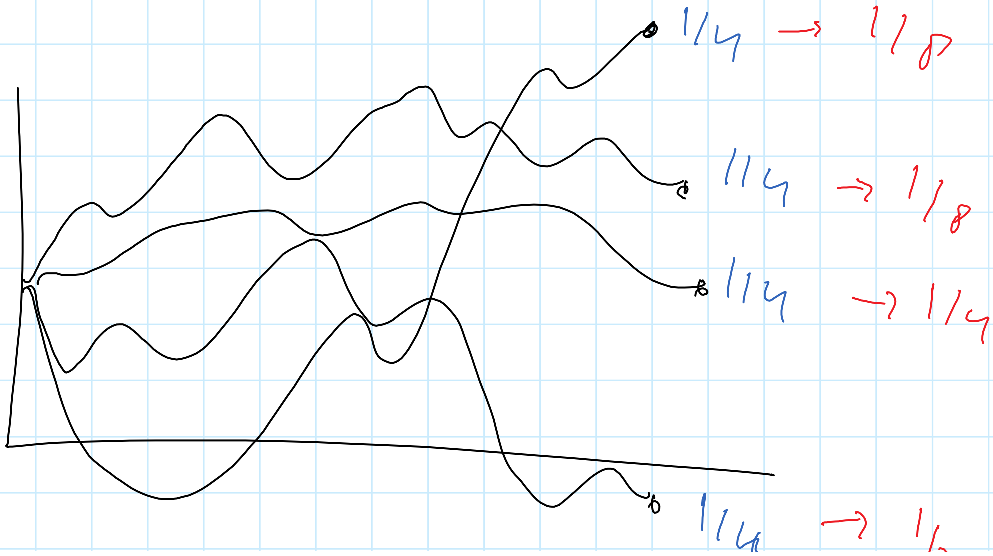
$\mathbb{P}^*$ – B.mtn.

$$X_t = x.$$



$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \mathcal{E}\left(\int_0^t \lambda_s \, dW_s \right)$$

$$W_t^* = W_t - \int_0^t \lambda_s \, ds \quad \text{is a } \mathbb{P}^* \text{– B.mtn.}$$



$1/4 \rightarrow 1/2$

no arb $\Longleftrightarrow$ $\exists$ $\mathbb{Q} \sim \mathbb{P}$ s.t.

relative prices of all traded assets

are martingales, i.e.

$(s > t)$ $$\frac{A_t}{B_t} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{A_s}{B_s}\right]$$ B is a numeraire $B > 0$ a.s.

suppose $\exists$ t s.t. and α (self-financing)

$V_t = \alpha_t \cdot S_t$

i) $\mathbb{P}(V_t \geq 0) = 1$

ii) $\mathbb{P}(V_t > 0) > 0$

and that $\exists$ $\mathbb{Q} \sim \mathbb{P}$ s.t. $\frac{S_t^{(k)}}{B_t} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{S_u^{(k)}}{B_u}\right]$

$\forall u > t, k$

since $B_t > 0$ ... since $\mathbb{Q} \sim \mathbb{P}$

i) $\Rightarrow$ $\mathbb{P}\left(\frac{V_t}{B_t} \geq 0\right) = 1 \Rightarrow \mathbb{Q}\left(\frac{V_t}{B_t} \geq 0\right) = 1$ $(*)$

ii) $\Rightarrow$ $\mathbb{P}\left(\frac{V_t}{B_t} > 0\right) > 0 \Rightarrow \mathbb{Q}\left(\frac{V_t}{B_t} > 0\right) > 0$ $(**)$

$$\frac{V_0}{B_0} = \cancel{\sum \alpha_0^{(k)} \frac{S_0^{(k)}}{B_0}}$$

$$\cancel{= \sum \alpha_0^{(k)} \mathbb{E}_0^{\mathbb{Q}}\left[\frac{S_t^{(k)}}{B_t}\right]}$$

$$\left. \qquad B_t \qquad \right]$$

$$= \mathbb{E}_0^{\mathbb{Q}}\left[\frac{V_t}{B_t}\right] > 0$$

↳ b.c. of (*) & (**)

$\Rightarrow \quad V_0 > 0 \quad \therefore$ there is no arbitrage.

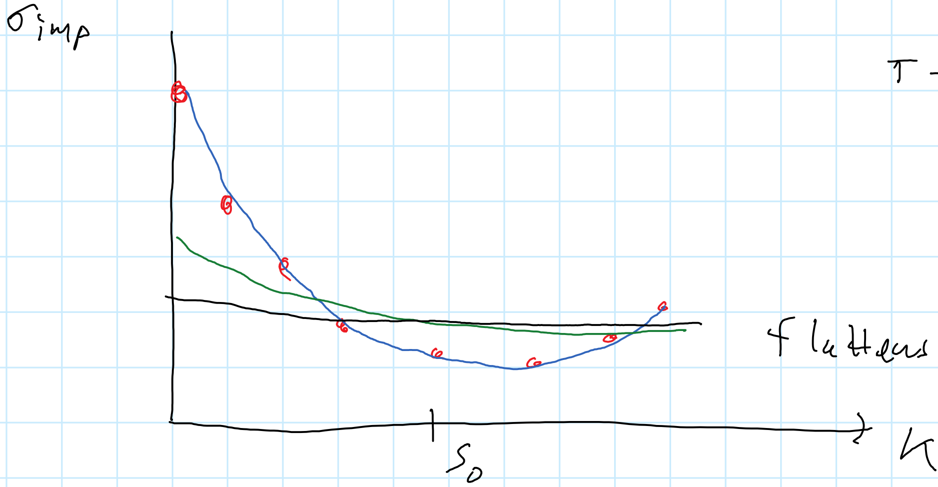
$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$
$$= r S_t \, dt + \sigma S_t \, dW_t^*$$

Wednesday, October 28, 2015 2:56 PM

$$C^{BS}(S_0, K; T, \sigma, r) = \ldots$$

$$\sigma^{\mathbb{P}} = \sigma^{\mathbb{Q}}$$

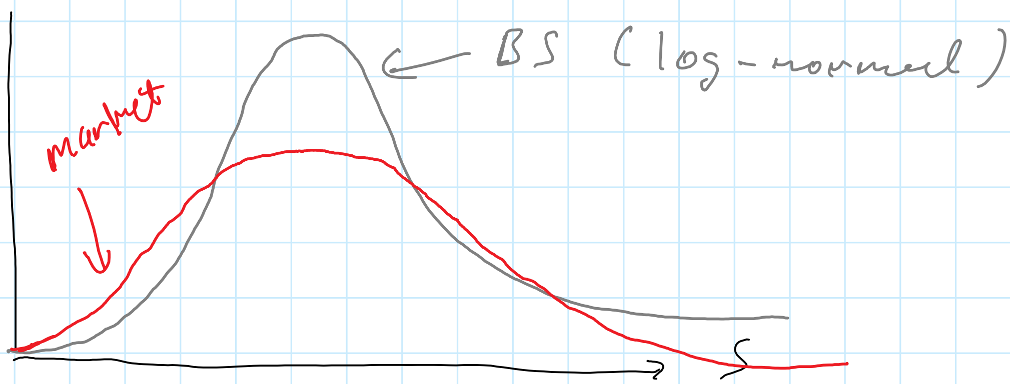
$$C^{M}(S_0, K; T, r) = C^{BS}(S_0, K; T, \sigma_{imp}(K,T), r)$$



T - fixed

implied volatility smile / smirk / skew.

flattens as $T \uparrow$.



models:

i) local volatility model

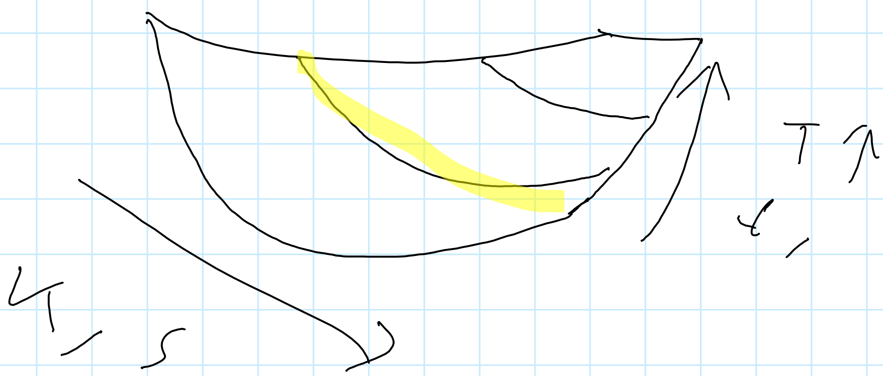
$$dS_t = \mu S_t \, dt + \sigma(t, S_t) \, S_t \, dW_t$$

$$= r S_t \, dt + \sigma(t, S_t) \, S_t \, dW_t^*$$

$\sigma \searrow$ as $S \nearrow$
(leverage effect)

$$\sigma^{imp}(K, T) \Longrightarrow \sigma^{local}(t, S)$$

Dupire's
equation



ii) stochastic volatility
(Heston model)

instantaneous
variance process

$$dS_t = \mu S_t \, dt + \sqrt{v_t} \, S_t \, dW_t$$

$$= r S_t \, dt + \sqrt{v_t} \, S_t \, dW_t^*$$

$$dv_t = \bar{\kappa}(\bar{\theta} - v_t) \, dt + \eta \sqrt{v_t} \, dB_t$$

$$= \kappa(\theta - v_t) \, dt + \eta \sqrt{v_t} \, dB_t^*$$

$$d[B, W]_t = d[B^*, W^*]_t = \rho \, dt$$

$$= \kappa(\theta - v_t)\,dt + \eta\sqrt{v_t}\,dB_t^*$$

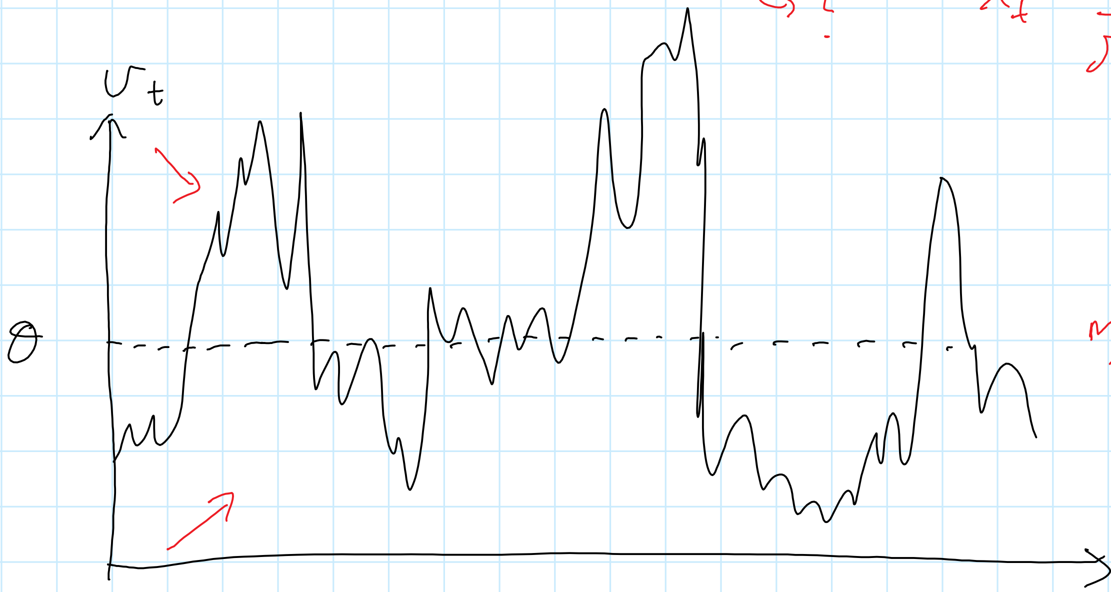
$$d[B,W]_t = d[B^*, W^*]_t = \rho\,dt$$

SDE is a Feller process ($2\bar{\kappa}\bar{\theta} > \eta^2$, $2\kappa\theta > \eta^2$)

$$B_t^* = B_t + \int_0^t \lambda_s\,ds$$

?

$$\lambda_t = \frac{\theta}{\sqrt{v_t}} + \circ\sqrt{v_t}$$



$$dv_t = \underbrace{\kappa(\theta - v_t)}_{\text{rate of m.r.}}\,dt + \underbrace{\eta\sqrt{v_t}}_{\text{vol-vol}}\,dB_t^*$$

$$\frac{P_t}{B_t} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{P_T}{B_T}\right] \quad \text{for a call}$$

$$\Rightarrow P_t = \mathbb{E}_t^{\mathbb{Q}}\left[(S_T - K)_+\, e^{-r(T-t)}\right]$$

$$= S_t\,\mathbb{Q}^S(S_T > K) - K e^{-r(T-t)}\,\mathbb{Q}(S_T > K)$$

$$= g(t, v_t, S_t)$$

due to Markov ppt. $\exists\ g(t, v, S)$ s.t.

$$\begin{cases} (\partial_t + \mathcal{L}^{S,v})\, g = r\, g \\ g(T, v, S) = (S - K)_+ \end{cases}$$

$$\mathcal{L}^{S,v} h(t, v, S) \triangleq \lim_{\varepsilon \searrow 0} \frac{\mathbb{E}^{\mathbb{Q}}_{t,v,S}\left[h(t, v_{t+\varepsilon}, S_{t+\varepsilon}) - h(t,v,S)\right]}{\varepsilon}$$

$$dS_t = r S_t\, dt + \sqrt{v_t}\, S_t\, dW_t^*$$

$$dv_t = \kappa(\theta - v_t)\, dt + \eta\sqrt{v_t}\, dB_t^*$$

$$\mathcal{L}^{S,v} = r S\, \partial_S + \tfrac{1}{2} v S^2 \partial_{SS}$$
$$+ \kappa(\theta - v)\, \partial_v + \tfrac{1}{2}\eta^2 v\, \partial_{vv}$$
$$+ \rho \eta v S\, \partial_{vS}$$

Heston is an affine model

$$X_t = \log S_t = f(S_t), \quad f(S) = \log S$$

$$dX_t = \Big(\underbrace{\partial_t \log(S_t)}_{0} + r S_t \underbrace{\partial_S \log S_t}_{1/S_t} + \tfrac{1}{2} v_t S_t^2 \underbrace{\partial_{SS} \log S_t}_{-1/S_t^2} \Big) dt$$

$$+ \sqrt{v_t}\, S_t \underbrace{\partial_s \log S_t}_{1/S_t} dW_t^*$$

$$= \left(r - \tfrac{1}{2} v_t\right) dt + \sqrt{v_t}\, dW_t^*$$

$$\mathcal{L}^{x,v} = \left(r - \tfrac{1}{2}v\right)\partial_x + \tfrac{1}{2} v\, \partial_{xx}$$

$$+ \kappa(\theta - v)\partial_v + \tfrac{1}{2}\eta^2 v\, \partial_{vv} + \rho\eta v\, \partial_{xv}$$

characteristic function can be computed in closed-form!

$$h(t,v,x) = \mathbb{E}_{t,v,x}\left[e^{iuX_T}\right]$$

$$\begin{cases} (\partial_t + \mathcal{L}^{x,v})\, h = 0 \\ \qquad h(T,v,x) = e^{iux} \end{cases}$$

$$h(t,v,x) = e^{A(t) + B(t)\,v + C(t)\,x}$$

$$A(T) = B(T) = 0\,, \quad C(T) = iu$$

$$(\partial_t + \mathcal{L}^{x,v})\, h$$

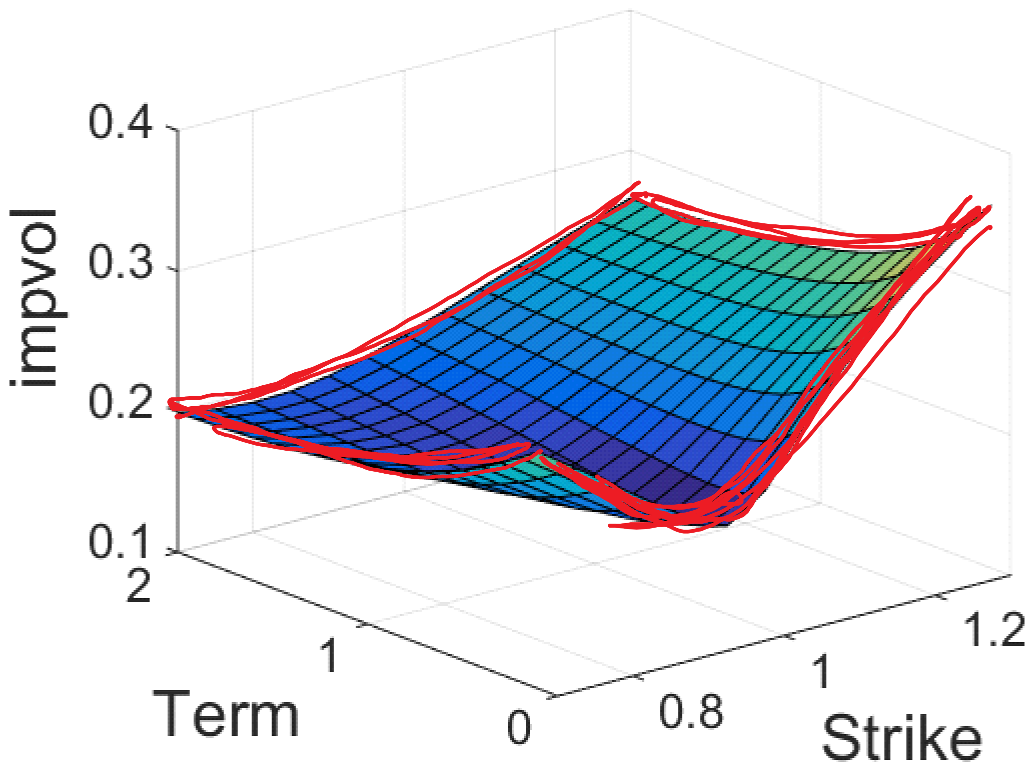
$$= \underbrace{(\quad)}_{0} h + \underbrace{(\quad)}_{0} v h + \underbrace{(\quad)}_{0} x h = 0$$

b/c must hold $\forall\, t, x, v$.

· leads to Riccati ODEs that are

explicitly solvable.

- use Fourier techniques to invert and compute prices.


impvol
0.4
0.3
0.2
0.1
2
1
0
Term
0.8
1
1.2
Strike