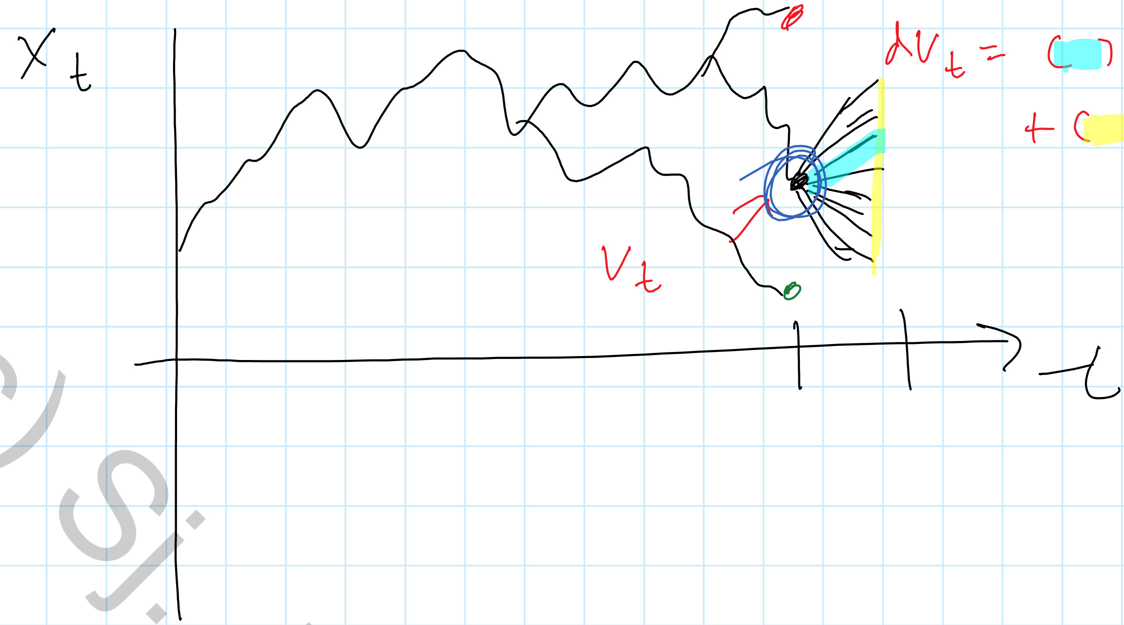


$dV_t = (\ \ )\,dt$
$+ (\ \ )\,dW_t$

when X is traded...

Wednesday, October 7, 2015 2:41 PM

$$\begin{cases} (\partial_t + \mathcal{L}) f(t,x) = r(t,x) f(t,x) \\ f(T,x) = F(x) \end{cases}$$

$$\mathcal{L} = r(t,x)\, x\, \partial_x + \tfrac{1}{2} (\sigma^X(t,x))^2 \partial_{xx}$$

Black-Scholes model assumes that an asset price $S = (S_t)_{t \geq 0}$ satisfies the SDE

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$$

Geometric Brownian motion (GBM)

and the short-rate of interest is constant:

$$r_t = r = \text{const.}$$

$$\mathcal{L} = r x \partial_x + \tfrac{1}{2} \sigma^2 x^2 \partial_{xx}$$

$$\begin{cases} (\partial_t + \mathcal{L}) f(t,x) = r \cdot f(t,x) \\ f(T,x) = F(x) \end{cases}$$

Black-Scholes PDE

---

e.g. $F(x) = 1$ i.e. a bond

guess that $f(t,x)$ is fn. only of t ...

guess that $f(t,x)$ is fn. only of t ...

$\Rightarrow \quad \mathcal{L} f = 0$ and so

$$\partial_t f = r f, \quad f(T,x) = 1$$

$$\Rightarrow \quad f(t,x) = e^{-r(T-t)}$$

e.g. $F(x) = x$ i.e. the stock itself

check is $f(t,x) = x$ a sol to the PDE?

$$\partial_t f = 0, \quad \partial_x f = 1, \quad \partial_{xx} f = 0$$

$$(\partial_t + \mathcal{L}) f = r f$$

$$(\underbrace{\partial_t}_{0} + r x \underbrace{\partial_x}_{1} + \tfrac{1}{2} \sigma^2 x^2 \underbrace{\partial_{xx}}_{0}) f \overset{?}{=} r \underbrace{f}_{x}$$

lhs = rhs and b.c. is satisfied

$f(T,x) = x = F(x)$.

e.g., $F(x) = a . x^2$, (power call with $K=0$)

$f(t,x) = h(t)\, x^2$ ansatz

$$\mathcal{L} = r x \partial_x + \tfrac{1}{2} \sigma^2 x^2 \partial_{xx}$$

$$\mathcal{L} f = h(t) \left\{ r x . 2x + \tfrac{1}{2} \sigma^2 . x^2 . 2 \right\}$$

$$= h(t) \left(r + \tfrac{1}{2} \sigma^2 \right) 2 x^2$$

$$(\partial_t + \mathcal{L}) f = r f$$

$$(\partial_t + \mathcal{L}) f = r f$$

$$\Rightarrow \quad \partial_t h(t) x^2 + h(t) \left(r + \tfrac{1}{2}\sigma^2\right) 2x^2 = r\, h(t) x^2$$

$$\Big[\underbrace{\partial_t h(t) + h(t)\,(r + \sigma^2)}_{=0} \Big] x^2 = 0$$

$$\Rightarrow \quad \partial_t h + h\,(r + \sigma^2) = 0$$

$$\& \quad h(T) = a$$

$$\Rightarrow \quad h(t) = a\, e^{(r + \sigma^2)(T - t)}$$

$$f(t, x) = a\, e^{(r + \sigma^2)(T - t)}\, x^2$$

Wednesday, October 7, 2015 3:23 PM

want to be able to solve PDEs:

$$\begin{cases} \left(\partial_t + a(t,x)\,\partial_x + \tfrac{1}{2} b^2(t,x)\,\partial_{xx}\right) f(t,x) = c(t,x)\, f(t,x) \\ f(T,x) = F(x) \end{cases}$$

The solution to the PDE:

$$\begin{cases} \left(\partial_t + \tfrac{1}{2}\partial_{xx}\right) f(t,x) = 0, \\ f(T,x) = F(x). \end{cases}$$

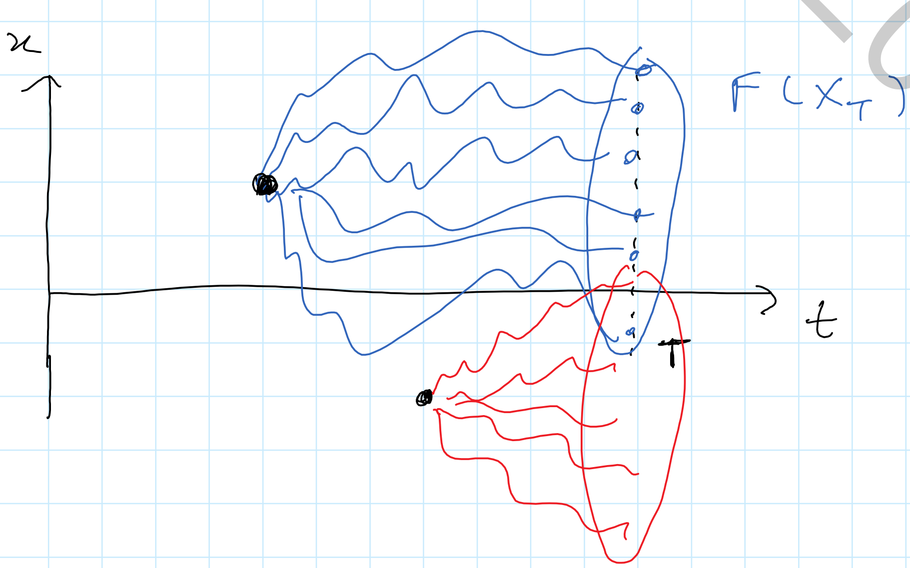
admits a stochastic representation:

$$f(t,x) = \mathbb{E}^{\mathbb{P}^*}_{t,x}\left[\, F(X_T)\,\right]$$

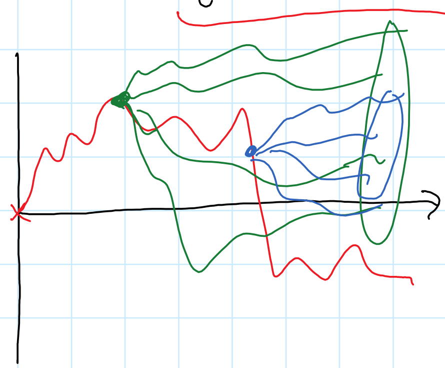
where $X = (X_t)_{t\geq 0}$ is a $\mathbb{P}^*$-B.mtn.

$$\mathbb{E}^{\mathbb{P}^*}_{t,x}[\;\cdot\;] = \mathbb{E}^{\mathbb{P}^*}[\;\cdot \mid X_t = x]$$

$$f(t,x) = \mathbb{E}^{\mathbb{P}^*}\left[\, F(X_T^{t,x}) \mid \mathcal{F}_t \right]$$



Proof: let $h_t \triangleq f(t, X_t)$, $h = (h_t)_{t \ge 0}$



$$h_t = \mathbb{E}_t^{\mathbb{P}^*}[F(X_T)] = \mathbb{E}^{\mathbb{P}^*}[F(X_T) \mid \mathcal{F}_t^X]$$

claim: h is a martingale.

$$\mathbb{E}_s^{\mathbb{P}^*}[h_t] = \mathbb{E}^{\mathbb{P}^*}[h_t \mid \mathcal{F}_s^X] \overset{?}{=} h_s \qquad (s < t)$$

$$= \mathbb{E}^{\mathbb{P}^*}\left[\mathbb{E}^{\mathbb{P}^*}[F(X_T) \mid \mathcal{F}_t^X] \mid \mathcal{F}_s^X\right]$$

$$= \mathbb{E}^{\mathbb{P}^*}[F(X_T) \mid \mathcal{F}_s^X]$$

$$= h_s$$

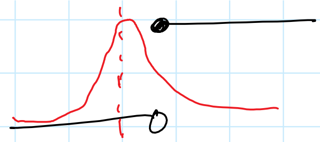
so now: $\mathbb{E}^{\mathbb{P}^*}[h_{t+\varepsilon} - h_t \mid \mathcal{F}_t] = 0$

$$h_t = \mathbb{E}_t^{\mathbb{P}^*}[F(X_T)]$$

$$= \mathbb{E}_t^{\mathbb{P}^*}[F(\underbrace{(X_T - X_t)}_{\sim \mathcal{N}(0, T-t)} + X_t)]$$

$$= \int_{-\infty}^{\infty} F(\sqrt{\tau}\, z + X_t)\, e^{-\frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}}$$

$$= h(t, X_t) \in C^{1,2}$$



$$h_{t+\varepsilon} - h_t = \int_t^{t+\varepsilon} \left(\partial_t + \tfrac{1}{2}\partial_{xx}\right) h(u, X_u)\, du + \int_t^{t+\varepsilon} \partial_x h(u, X_u)\, dW_u$$

$$\left(dh_t = \left(\partial_t + \tfrac{1}{2}\partial_{xx}\right) h(t, X_t)\, dt + \partial_x h(t, X_t)\, dW_t \right)$$

$X_t = x$

apply $\mathbb{E}_{t,x}[\ \cdot\ ]$ to both sides

mtg.

$$0 = \mathbb{E}_{t,x}\left[\int_t^{t+\varepsilon} \left(\partial_t + \tfrac{1}{2}\partial_{xx}\right) h(u, X_u)\, du\right]$$

$$\Rightarrow \quad 0 = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}_{t,x}\left[\int_t^{t+\varepsilon} \left(\partial_t + \tfrac{1}{2}\partial_{xx}\right) h(u, X_u)\, du\right]$$

F.T.C.

$$= \mathbb{E}_{t,x}\left[\left(\partial_t + \tfrac{1}{2}\partial_{xx}\right) h(t, X_t) \right]$$

$$= \left(\partial_t + \tfrac{1}{2}\partial_{xx}\right) h(t, x)$$

$$\begin{cases} \left(\partial_t + \tfrac{1}{2}\partial_{xx}\right) h(t, x) = 0 \\ h(T, x) = \mathbb{E}^{\mathbb{P}^\alpha}_{T,x}[\, F(X_T)\,] = F(x) \end{cases}$$

hence $F(t,x) = \mathbb{E}^{\mathbb{P}^*}_{t,x}[\, F(X_T)]$
satisfies the required PDE. $\boxtimes$

Feynman-Kac Theorem

Wednesday, October 7, 2015 4:19 PM

The solution to

$$\begin{cases} \left(\partial_t + a(t,x)\,\partial_x + \tfrac{1}{2} b^2(t,x)\,\partial_{xx}\right) f(t,x) = c(t,x)\, f(t,x) \\ f(T,x) = F(x) \end{cases}$$

admits a stochastic representation

$$f(t,x) = \mathbb{E}^{\mathbb{P}^*}_{t,x}\left[e^{-\int_t^T c(u,X_u)\,du}\; F(X_T) \right]$$

where $X = (X_t)_{t\geq 0}$ is stochastic process satisfying the SDE:

$$dX_t = a(t,X_t)\,dt + b(t,X_t)\,dW^*_t$$

where $W^* = (W^*_t)_{t\geq 0}$ is a $\mathbb{P}^*$ - B. mtn.

For generalized B-S PDE:

$$a(t,x) = \mu^x(t,x) - \sigma^x(t,x)\,\lambda(t,x)$$

$(r(t,x)\,x$ when X is traded$)$

$$b(t,x) = \sigma^x(t,x)$$

$$c(t,x) = r(t,x)$$

recall that:

$\mathbb{P}$ - B. mtn.

$$dX_t = \mu^x(t,X_t)\,dt + \sigma^x(t,X_t)\,dW_t$$

for financial models.

modelling

$$dX_t = \left(\mu^X(t,X_t) - \sigma^X(t,X_t)\,\lambda(t,X_t) \right) dt + \sigma^X(t,X_t)\, dW_t^*$$

to solve PDE

$\mathbb{P}^*$ - B.mtn

what if $\quad dW_t^* = \lambda(t,X_t)\,dt + dW_t$

i.e. $\quad W_t^* = \int_0^t \lambda(u,X_u)\,du + W_t$

Girsanov's Thm:

Let $W = (W_t)_{0 \le t \le T}$ be a $\mathbb{P}$ - B.mtn

& define
$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \mathcal{E}\left(\int_0^T \lambda_u \, dW_u \right)$$

Doleans - Dade exponential (stochastic exponential)

then,
$$W_t^* = -\int_0^t \lambda_u \, du + W_t$$

is a $\mathbb{P}^*$ - B.mtn.

Doleans - Dade Exponential:

$$\eta = \mathcal{E}\left(\int_0^T \lambda_u \, dW_u \right)$$

$\eta_t \triangleq \mathbb{E}_t[\,\eta\,]$ and solves

$$d\eta_t = \lambda_t \, \eta_t \, dW_t \quad , \quad \eta_0 = 1$$

$$\eta_t = \exp\left\{ -\frac{1}{2}\int_0^t \lambda_u^2 \, du + \int_0^t \lambda_u \, dW_u \right\}$$

$F_t = f(\eta_t)$, $f : \mathbb{R} \mapsto \mathbb{R}$
$x \mapsto \log x$

$$dF_t = (\partial_t + \mathcal{L}) f(\eta_t)\, dt + \partial_x f(\eta_t) \cdot \lambda_t \eta_t \cdot dW_t$$

$$\left(\begin{array}{l} \partial_t f = 0 , \quad \partial_x f = \dfrac{1}{x} , \quad \partial_{xx} f = -\dfrac{1}{x^2} \\ \mathcal{L} = 0 \cdot \partial_x + \dfrac{1}{2} \cdot \lambda^2 x^2 \, \partial_{xx} \end{array} \right)$$

$$= + \frac{1}{2} \lambda_t^2 \cancel{\eta_t^2} \cdot \left(- \frac{1}{\cancel{\eta_t^2}} \right) dt$$

$$+ \frac{1}{\cancel{\eta_t}} \cdot \lambda_t \cancel{\eta_t} \, dW_t$$

$$= -\frac{1}{2} \lambda_t^2 \, dt + \lambda_t \, dW_t$$

$$\log \eta_t - \log \underbrace{\eta_0}_{=1} = -\frac{1}{2}\int_0^t \lambda_u^2 \, du + \int_0^t \lambda_u \, dW_u$$

$$\Rightarrow \quad \eta_t = e^{-\frac{1}{2}\int_0^t \lambda_u^2 \, du + \int_0^t \lambda_u \, dW_u}$$

$$Z \overset{\mathbb{P}}{\sim} N(0,1)$$

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left\{ -\tfrac{1}{2}\lambda^2 + \lambda Z \right\}$$

$$\left(\text{1she} \quad Z \overset{\mathbb{P}^*}{\sim} N(\cdot \, , \, 1) \right)$$

$$\mathbb{E}^{\mathbb{P}^*}\left[e^{iuZ} \right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[e^{iuZ} \frac{d\mathbb{P}^*}{d\mathbb{P}} \right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[e^{iuZ} \, e^{-\frac{1}{2}\lambda^2 + \lambda Z} \right]$$

$$= e^{-\frac{1}{2}\lambda^2} \, \mathbb{E}^{\mathbb{P}}\left[e^{(iu+\lambda)Z} \right]$$

$$= e^{-\frac{1}{2}\lambda^2} \, e^{\frac{1}{2}(iu+\lambda)^2}$$

$$= e^{-\frac{1}{2}u^2 + iu\lambda}$$

is characteristic fn of

$$N(\lambda; 1)$$

$$Z \overset{\mathbb{P}^*}{\sim} N(\lambda; 1)$$

$$z^* = z - \lambda \overset{\mathbb{P}^*}{\sim} \mathcal{N}(0,1)$$