Discrete time model

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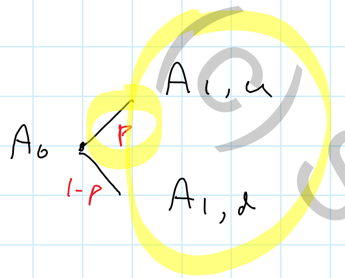
asset $(A_t)_{t \in \{0,1,\ldots\}}$ $\hookrightarrow \{0,1\}$

$A_0 = \cancel{A}$, $\quad A_1 = A_{1,u}\, x + A_{1,d}\,(1-x)$

Bernoulli r.v.

$\mathbb{P}(x=0) = 1-p$

$\mathbb{P}(x=1) = p$



$$A_0 = \frac{\mathbb{E}[A_1]}{1+r} = \frac{p\, A_{1,u} + (1-p)\, A_{1,d}}{1+r}$$

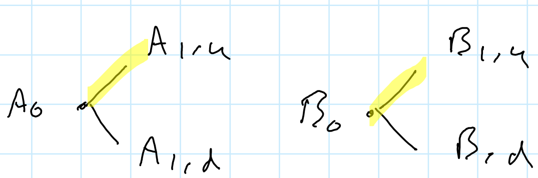
× utility theory

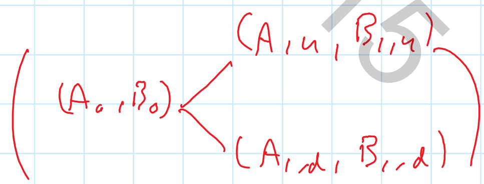
× prospect theory

another asset $(B_t)_{t \in \{0,1,\ldots\}}$

$B_0 = \cancel{B}$

$B_1 = B_{1,u}\, x + B_{1,d}\,(1-x)$





An arbitrage portfolio (α, β) is one s.t.

i) $V_0 = \alpha A_0 + \beta B_0 = 0$ (value of the portfolio at $t=0$)

(cost nothing)

ii) $\mathbb{P}(V_1 \geq 0) = 1$ (never lose)

iii) $\mathbb{P}(V_1 > 0) > 0$ (sometimes win)

$0 \left\langle \begin{matrix} 1 \\ -1 \end{matrix} \right.$ not an arb $\qquad 0 \left\langle \begin{matrix} 1 \\ 0 \end{matrix} \right.$ is an arb.

$0 \left\langle \begin{matrix} 0 \\ 0 \end{matrix} \right.$ not an arb $\qquad 1 \left\langle \begin{matrix} 1 \\ 2 \end{matrix} \right. + -1 \left\langle \begin{matrix} -1 \\ -1 \end{matrix} \right.$

is not an arb.

$= 0 \left\langle \begin{matrix} 0 \\ 1 \end{matrix} \right.$

$$1.5 \left\langle \begin{matrix} 1 \\ 2 \end{matrix} \right. + \; -0.5 \left\langle \begin{matrix} -1 \\ -1 \end{matrix} \right.$$

$1 \qquad \times 3$

$$= \; 0 \left\langle \begin{matrix} -2 \\ -1 \end{matrix} \right.$$

$$1.25 \left\langle \begin{matrix} 1 \\ 2 \end{matrix} \right. \qquad -0.75 \left\langle \begin{matrix} -1 \\ -1 \end{matrix} \right.$$

$$A_0 \left\langle \begin{matrix} A_u \\ A_d \end{matrix} \right. \qquad B_0 \left\langle \begin{matrix} B_u \\ B_d \end{matrix} \right.$$

$\alpha \qquad \beta$

$$A_0\,\alpha + B_0\,\beta = \left(A_u - B_u \cdot \frac{A_0}{B_0}\right) \cdot \alpha$$

$$A_0\alpha + B_0\beta = 0 \quad \begin{cases} A_u\alpha + B_u\beta = \left(A_u - B_u \cdot \frac{A_0}{B_0}\right)\cdot\alpha \\ A_d\alpha + B_d\beta = \left(A_d - B_d \cdot \frac{A_0}{B_0}\right)\alpha \end{cases}$$

$$\Downarrow$$

$$\beta = -\frac{A_0}{B_0}\alpha$$

$$\left.\begin{array}{l} \left(A_u - B_u \cdot \frac{A_0}{B_0}\right)\alpha > 0 \\ \text{and} \quad \left(A_d - B_d \cdot \frac{A_0}{B_0}\right)\alpha < 0 \end{array}\right\} \Rightarrow \text{not an arb.}$$

$$\alpha > 0 \Rightarrow \quad A_u - \frac{B_u}{B_0}A_0 > 0 \quad (1)$$

$$A_d - \frac{B_d}{B_0}\cdot A_0 < 0$$

$(B_u, B_d > 0)$ $\Rightarrow$
numeraire asset

$$(1) \Rightarrow \quad \frac{A_u}{B_u} > \frac{A_0}{B_0}$$

$$(2) \Rightarrow \quad \frac{A_d}{B_d} < \frac{A_0}{B_0}$$

$$\boxed{\frac{A_d}{B_d} \underset{>}{<} \frac{A_0}{B_0} \underset{>}{<} \frac{A_u}{B_u}} \iff \text{absence of arbitrary}$$

e.g.

$$B = 1 \begin{cases} 1+r \\ 1+r \end{cases} \qquad \frac{A_d}{1+r} < A_0 < \frac{A_u}{1+r}$$

$$A_d < A_0(1+r) < A_u$$

relative price $\tilde{A}_t = \dfrac{A_t}{B_t}$

$$\frac{A_0}{B_0} \begin{cases} \dfrac{A_u}{B_u} \\[2ex] \dfrac{A_d}{B_d} \end{cases}$$

$\dfrac{A_d}{B_d}$ $\dfrac{A_0}{B_0}$ $\dfrac{A_u}{B_u}$

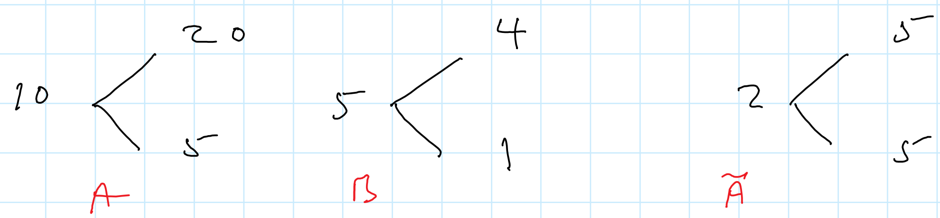
$\exists \; q \in (0,1)$ s.t. $\dfrac{A_0}{B_0} = \dfrac{A_u}{B_u} q + \dfrac{A_d}{B_d} \cdot (1-q)$

$$\Rightarrow q = \frac{\dfrac{A_0}{B_0} - \dfrac{A_d}{B_d}}{\dfrac{A_u}{B_u} - \dfrac{A_d}{B_d}}$$

$$\frac{A_0}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_1}{B_1} \right]$$

$\exists$ no arbitrage in the market if and only if $\exists$ a $\mathbb{Q} \sim \mathbb{P}$ (and a numeraire B) s.t.

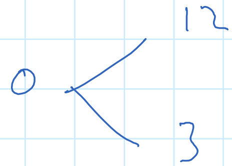
$$\tilde{A}_0 = \mathbb{E}^{\mathbb{Q}}[\tilde{A}_1]$$ (for all traded assets A)

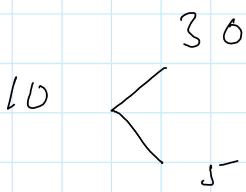


an arbitrage economy!

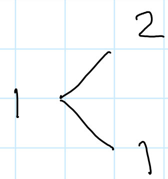
$\times 1$ $\times -2$

12

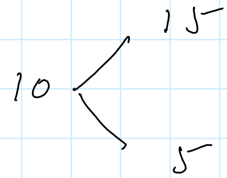




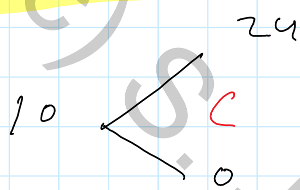
A



B



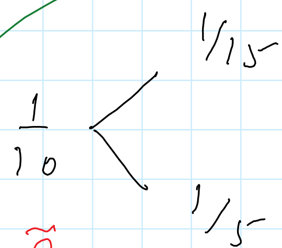
$\tilde{A}$



$\exists\, q^B,\quad q^B = 1/2,$ s.t.

$$\tilde{A}_0 = \mathbb{E}^{\mathbb{Q}^B}[\tilde{A}_1]$$

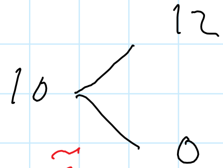
$\Rightarrow$ economy is arbitrage free



$\tilde{B}$

$\exists\, q^A \in (0,1)$ s.t.

$$\tilde{B}_0 = \mathbb{E}^{\mathbb{Q}^A}[\tilde{B}_1]$$



$\tilde{C}$

$\exists\, q^* \in (0,1)$ s.t.

$$\tilde{C}_0 = \mathbb{E}^{\mathbb{Q}^*}[\tilde{C}_1]$$

$$10 = 12\, q^* + 0\,(1-q^*)$$

$$\Rightarrow q^* = \frac{10}{12} = \frac{5}{6} \neq q^B$$

¿ $\tilde{A}_0 = \mathbb{E}^{\mathbb{Q}^*}[\tilde{A}_1]$?

$$70 \neq 15 \cdot \frac{5}{6} + 5 \cdot \frac{1}{6}$$

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$$\frac{A_0}{B_0} = \mathbb{E}^{\mathbb{Q}^B}\left[\frac{A_1}{B_1}\right] \qquad \forall \text{ traded assets } A$$

$$\frac{A_0}{C_0} = \mathbb{E}^{\mathbb{Q}^C}\left[\frac{A_1}{C_1}\right] \qquad \forall \text{ traded assets } A$$

$$\frac{A_0}{C_0} = \frac{A_u}{C_u}\underbrace{\left(\frac{C_u/C_0}{B_u/B_0}\, q^B\right)}_{\to q^*} + \frac{A_d}{C_d}\underbrace{\left(\frac{C_d/C_0}{B_d/B_0}\,(1-q^B)\right)}_{q^{**}}$$

$$\frac{A_0}{C_0} = \frac{A_u}{C_u}\, q^C + \frac{A_d}{C_d}\,(1-q^C)$$

¿ $q^C = \dfrac{C_u/C_0}{B_u/B_0}\, q^B$?

$$q^* + q^{**} = \frac{C_u/C_0}{B_u/B_0}\cdot q^B + \frac{C_d/C_0}{B_d/B_0}\,(1-q^B)$$

$$= \frac{B_0}{C_0}\left[\frac{C_u}{B_u}\, q^B + \frac{C_d}{B_d}\,(1-q^B)\right]$$

$$= \frac{B_0}{C_0}\,\underbrace{\mathbb{E}^{\mathbb{Q}^B}\left[\frac{C_1}{B_1}\right]}_{C_0/B_0} = 1$$

$$q^C = \frac{C_u/C_0}{B_u/B_0}\cdot q^B$$

$$q^C = \frac{C_u / C_0}{B_u / B_0} \cdot q^B$$

$$1 - q^C = \frac{C_d / C_0}{B_d / B_0} \cdot (1 - q^B)$$

$$\mathbb{Q}^C(\cdot) = \boxed{\left(\frac{C_1 / C_0}{B_1 / B_0}\right)(\cdot)} \cdot \mathbb{Q}^B(\cdot)$$

$$\frac{d\mathbb{Q}^C}{d\mathbb{Q}^B} = \frac{C_1 / C_0}{B_1 / B_0}$$

Radon-Nikodym derivative

$$10 \begin{array}{l} \nearrow 20 \\ \searrow 5 \end{array} \qquad 4 \begin{array}{l} \nearrow 10 \\ \searrow 1 \end{array} \qquad C_0 = ? \begin{array}{l} \nearrow 10 \\ \searrow 5 \end{array}$$

A B C

$$\frac{5}{2} \begin{array}{l} \nearrow 2 \\ \searrow 5 \end{array} \quad \Rightarrow \quad q^B = \#$$

$$\frac{C_0}{4} = \frac{10}{10} \cdot q^B + \frac{5}{1} \cdot (1 - q^B)$$

replication

$$A_0 \left\langle \begin{array}{l} A_u \\ A_d \end{array} \right. \qquad B_0 \left\langle \begin{array}{l} B_u \\ B_d \end{array} \right. \qquad C_0 \left\langle \begin{array}{l} C_u \\ C_d \end{array} \right.$$

$$\alpha \qquad\qquad \beta$$

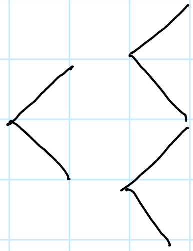
$$C_0 = \alpha A_0 + \beta B_0 \left\langle \begin{array}{l} \alpha A_u + \beta B_u = C_u \\ \alpha A_d + \beta B_d = C_d \end{array} \right\} \Rightarrow \begin{array}{l} \alpha = \ldots \\ \beta = \ldots \end{array}$$

$$= B_0 \cdot \left[\frac{C_u}{B_u} \cdot \left(q^B \right) + \frac{C_d}{B_d} \left(1 - q^B \right) \right]$$

$$q^B = \frac{\frac{A_0}{B_0} - \frac{A_d}{B_d}}{\frac{A_u}{B_u} - \frac{A_d}{B_d}}$$

$\sigma\sqrt{\Delta t}$?

$$A_n = A_{n-1}\, e^{c\, x_n}$$



$x_1,\ x_2, \ldots\ldots\ (\pm 1)$