Credit Models

τ - time of default of firm.

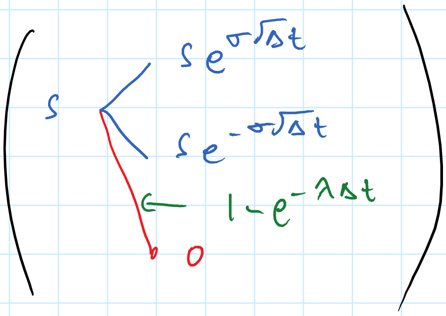
$$S_t = S_0\, e^{(\mu-\frac{1}{2}\sigma^2)t + \sigma W_t}$$

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t)$$

no default

$$S_t = S_0\, e^{(\mu-\frac{1}{2}\sigma^2)t + \sigma W_t} \quad \mathbb{1}_{\tau > t}$$

↳ 1st arrival of Poisson process (intensity λ) i.e. exponential r.v. mean 1/λ.



NB: $\mathbb{E}[S_T] = S_0 e^{\mu T}$ if no default (i.e. $\lambda=0$)

$\mathbb{E}[S_T] = S_0 e^{\mu T} \cdot \mathbb{P}(\tau > T) \rightarrow e^{-\lambda T}$

↑ assuming independence of τ & W

⇒ rate of return of S is $(\mu - \lambda)$

$e^{-rt} S_t$ is a $\mathbb{Q}$-mtg.

$$\mathbb{P} \xrightarrow[\text{Counting Process } (N_t)]{\text{B.mtn } (W_t)} \mathbb{Q}$$

can choose class of measures s.t. N_t is still Poisson $\hat{\lambda}$

$$\Rightarrow \quad S_t = S_0\, e^{(\mu-\frac{1}{2}\sigma^2)t + \sigma(\underbrace{\overset{\mathbb{Q}}{\hat{W}_t} + \gamma t}_{W_t})} \quad \mathbb{1}_{\tau > t}$$

↳ $\mathbb{P}$

W_t $\tau > t$

$\hookrightarrow \mathbb{P}$

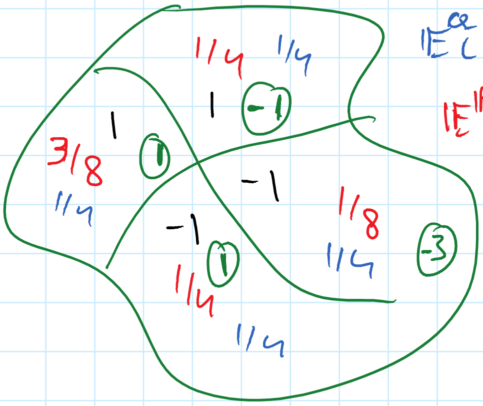
$Z \underset{\mathbb{P}}{\sim} N(0,1)$

$\mathbb{P} \longrightarrow \mathbb{Q}$

$Z \underset{\mathbb{Q}}{\sim} N(\gamma, 1)$

$Z = \hat{Z} + \gamma$

$\hat{Z} \underset{\mathbb{Q}}{\sim} N(0,1)$



$\mathbb{E}^{\mathbb{P}}[Z) = \frac{1}{4}$

$\mathbb{E}^{\mathbb{Q}}[Z] = 0$

$\mathbb{E}^{\mathbb{P}}[\hat{Z}] = 0$

$\mathbb{E}^{\mathbb{Q}}[\hat{Z}] = -\frac{1}{2}$

$$\mathbb{E}^{\mathbb{Q}}[S_T] = S_0 \, e^{(\mu + \gamma\sigma - \hat{\lambda})T}$$

$\mathbb{Q}$ is risk-neutral $\iff$ $r = \mu + \gamma\sigma - \hat{\lambda}$

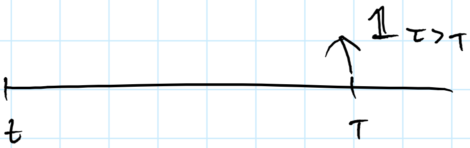
$$\gamma = \frac{r - \mu + \hat{\lambda}}{\sigma}$$

$$= -\left(\frac{\mu - (r + \hat{\lambda})}{\sigma}\right)$$

looks like Sharpe ratio

$r \to r + \hat{\lambda}$

Bond issue:

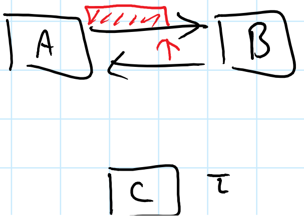


$$\Rightarrow \bar{P}_t(T) = \mathbb{E}^{\mathbb{Q}}[e^{-rT} \, \mathbb{1}_{\tau > T}]$$

$$= e^{-(r + \hat{\lambda})T} = (e^{-rT})(e^{-\hat{\lambda}T}) \rightarrow \mathbb{Q}(\tau > T)$$

$\hookrightarrow$ def. free bond

Credit Default Swaps

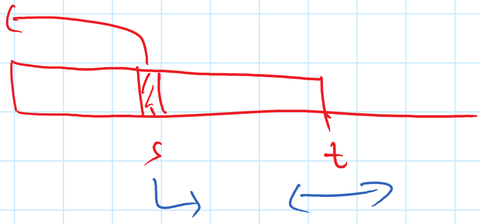


$$V^{prem} = \mathbb{E}^{\mathbb{Q}}\left[\int_0^{\tau \wedge T} F \cdot e^{-rs} \cdot ds\right]$$

$$V^{def} = \mathbb{E}^{\mathbb{Q}}[\underset{(1-R)}{1} \, e^{-r\tau} \, \mathbb{1}_{\tau < T}]$$

$\boxed{C}$ τ

$$V^{def} = \mathbb{E}^{\mathbb{Q}}\Big[\overset{\to(1-R)}{\textcircled{1.}}\, e^{-r\tau}\ \mathbb{1}_{\tau<T}\Big]$$



F = CDS rate set so that $V^{prem} = V^{def}$ at start.

$$V^{def} = (1-R)\int_0^{\infty} e^{-rs}\ \mathbb{1}_{s<T}\cdot\hat{\lambda}e^{-\hat{\lambda}s}\,ds$$

$$= (1-R)\hat{\lambda}\int_0^T e^{-(r+\hat{\lambda})s}\,ds$$

$$= (1-R)\hat{\lambda}\ \frac{1-e^{(r+\hat{\lambda})T}}{r+\hat{\lambda}}$$

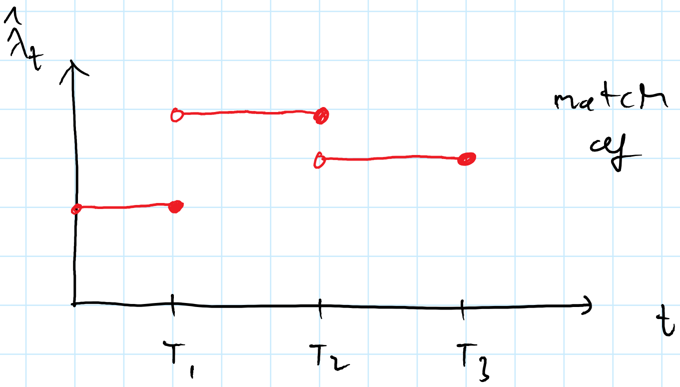
$$V^{prem} = F\,\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T e^{-rs}\ \mathbb{1}_{\tau>s}\,ds\Big]$$

$$= F\int_0^T e^{-rs}\cdot e^{-\hat{\lambda}s}\,ds$$

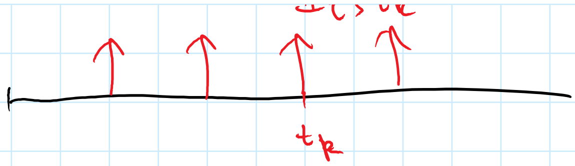
$$= F\int_0^T e^{-(r+\hat{\lambda})s}\,ds$$

$$= F\ \frac{1-e^{-(r+\hat{\lambda})T}}{r+\hat{\lambda}}$$

$$\boxed{F = (1-R)\hat{\lambda}}$$



match CDS rates of various maturities.


t_k

stochastic intensity

doubly stochastic Poisson process

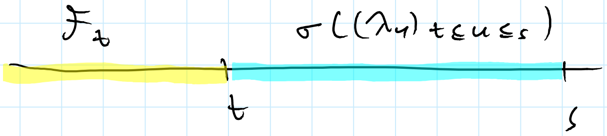
$(N_t, \lambda_t)_{0 \le t \le T}$ λ_t is the $\mathcal{F}_t$-conditional intensity of counting process N_t.

$$\mathbb{P}(N_{t+\Delta t} - N_t = 0 \mid \mathcal{F}_t) = (1 - \lambda_t \Delta t) + o(\Delta t)$$

$$\mathbb{P}(N_{t+\Delta t} - N_t = 1 \mid \mathcal{F}_t) = \lambda_t \Delta t + o(\Delta t)$$

$$\mathbb{P}(N_{t+\Delta t} - N_t \ge 2 \mid \mathcal{F}_t) = o(\Delta t)$$

$(N_u)_{t \le u \le s} \Big|_{\mathcal{F}_t \vee \sigma((\lambda_u)_{t \le u \le s})}$ is an inhomogeneous Poisson process intensity $(\lambda_u)_{t \le u \le s}$



$$\mathbb{P}(N_s - N_t = 0 \mid \mathcal{F}_t)$$

$$= \mathbb{E}[\mathbb{1}_{N_s - N_t = 0} \mid \mathcal{F}_t]$$

$$= \mathbb{E}\Big[\underbrace{\mathbb{E}[\mathbb{1}_{N_s - N_t = 0} \mid \mathcal{F}_t \vee \sigma((\lambda_u)_{t \le u \le s})]}_{e^{-\int_t^s \lambda_u du}} \Big| \mathcal{F}_t\Big]$$

$$= \mathbb{E}[e^{-\int_t^s \lambda_u du} \mid \mathcal{F}_t]$$

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e.g. $d\lambda_t = \kappa(\theta - \lambda_u)du + \sigma dW_t$ (OU)

$d\lambda_t = \kappa(\theta - \lambda_u)du + \sigma\sqrt{\lambda_t}\, dW_t$ (Feller)

$$\mathbb{E}[N_t] = \mathbb{E}[\mathbb{E}[N_t \mid \sigma((\lambda_u)_{0 \le u \le t})]]$$

$$= \mathbb{E}[\int_0^t \lambda_u du]$$

$$= \int_0^t \mathbb{E}[\lambda_u] du$$

$$= \int_0^t \mathbb{E}[\lambda_u]\, du$$

$\hat{N}_t = N_t - \int_0^t \lambda_u\, du$ is a $\mathbb{P}$-mtg !

defaultable bond $\qquad$ λ_t is $\mathbb{Q}$-intensity.

...

$$\bar{P}_t(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s\, ds}\, \mathbb{1}_{\tau > T} \,\middle|\, \mathcal{F}_t \right] \qquad (\lambda_u, r_u)_{t \le u \le T}$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\underline{e^{-\int_t^T r_s ds}}\, \mathbb{1}_{\tau > T} \,\middle|\, \underline{\mathcal{F}_t} \vee \sigma\left((\lambda_u)_{t \le u \le T}\right) \right] \middle|\, \mathcal{F}_t \right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \cdot e^{-\int_t^T \lambda_s ds} \,\middle|\, \mathcal{F}_t \right] \mathbb{1}_{\tau > t}$$

NB: $\mathbb{E}\left[\mathbb{1}_{\tau > T} \,\middle|\, \mathcal{F}_t \vee \sigma\left((\lambda_u)_{t \le u \le T}\right) \right]$

$$= e^{-\int_t^T \lambda_s ds}\, \mathbb{1}_{\tau > t}.$$

$$\Rightarrow \boxed{\bar{P}_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T (r_s + \lambda_s)\, ds} \right] \mathbb{1}_{\tau > t}}$$

$$\neq \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \right] \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \lambda_s ds} \right] \mathbb{1}_{\tau > t}$$

$$\frac{\bar{P}_t(T)}{P_t(T)} = \mathbb{E}_t^{\mathbb{Q}^T}\left[\frac{\mathbb{1}_{\tau > T}}{\underbrace{P_T(T)}_{1}} \right]$$

$$\Rightarrow \boxed{\bar{P}_t(T) = P_t(T) \cdot \mathbb{Q}_t^T(\tau > T)}$$

e.g.

$$d r_t = \kappa(\theta - r_t)\, dt + \sigma\, dW_t^r$$

$$d \lambda_t = \alpha(\phi - r_t - \lambda_t)\, dt + \eta\, dW_t^\lambda$$

corr = 0.

$$\left.\begin{array}{ll} r_t = \theta + x_t , & dx_t = -\alpha\, x_t\, dt + \sigma\, dW_t^x \\ \lambda_t = \phi + y_t & dy_t = -\beta\, y_t\, dt + \eta\, dW_t^y \end{array}\right\rangle \rho\, dt$$

def. free : $P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s\, ds} \right] = h(t, x_t)$

$$\Rightarrow \begin{cases} \partial_t h + \mathcal{L}^x h = (\theta + x)\, h \\ \underline{h(T,x) = 1} \end{cases}$$

$$\mathcal{L}^x = -\kappa x\, \partial_x + \tfrac{1}{2}\sigma^2\, \partial_{xx}$$

bond prices are affine since $\mathcal{L}^x$ is linear in x
& B.C. is e^0 , so,

$$h(t,x) = e^{A(t) - B(t)\, x} \quad , \quad \text{solve for } A \,\&\, B$$

$$\underline{A(T) = B(T) = 0}$$

$$\partial_t h = (\dot{A} - \dot{B}\, x)\, h , \quad \partial_x h = -B\, h, \quad \partial_{xx} h = B^2 h$$

$$\left\{ (\dot{A} - \dot{B} x) - \kappa x\, (-B) + \tfrac{1}{2}\sigma^2 B^2 \right\} h = (\theta + x)\, h$$

$$(\dot{A} + \tfrac{1}{2}\sigma^2 B^2 - \theta) + x\,(-\dot{B} + \kappa B - 1) = 0$$

must hold $\forall\ t, x$ $\Rightarrow \begin{cases} \dot{A} + \tfrac{1}{2}\sigma^2 B^2 - \theta = 0 \\ \dot{B} - \kappa B + 1 = 0 \end{cases} \Big\|$

$\longrightarrow$ solve to find A, B.

- what is $\mathbb{Q}^T$ dynamics of λ_t?

$$\eta_t = \left(\frac{d\mathbb{Q}^T}{d\mathbb{Q}} \right) \Bigg|_{\mathcal{F}_t} = \frac{P_t(T) / P_0(T)}{M_t \ / M_0}$$

$$\frac{d\eta_t}{\eta_t} = (\quad 0 \quad)\, dt + \cdots\, dW_t$$

recall $P_t(T) = e^{A_t - B_t x_t}$

$dx_t = -\alpha x_t\, dt + \sigma\, dW_t^x$

$$dP_t(T) = (\cdots)\, dt - B_t\, dx_t\, P_t(T)$$

$$\Rightarrow dP_t(T) = (\cdots)\, dt - \sigma B_t P_t(T)\, dW_t^x$$

so $\boxed{\frac{d\eta_t}{\eta_t} = -\sigma B_t\, dW_t^x}$

Girsanov Thm $\Rightarrow$

$$d\widehat{W}_t^x = \sigma B_t\, dt + dW_t^x$$

$$d\widehat{W}_t^y = \rho\sigma B_t\, dt + dW_t^y$$

$\mathbb{Q}^T$- B.mtn correlation ρ.

$$\therefore\quad dy_t = -\beta y_t\, dt + \eta\left(d\widehat{W}_t^y - \rho\sigma B_t\, dt\right)$$

$$= \beta\left(-\frac{\rho\sigma}{\beta} B_t - y_t\right) dt + \eta\, d\widehat{W}_t^y$$

we need $\mathbb{Q}_t^T(\tau > T) = \mathbb{E}_t^{\mathbb{Q}^T}\left[e^{-\int_t^T \lambda_s\, ds} \right] = g(t, y_t)$

$$\begin{cases} \partial_t g + \mathcal{L}^y g = (\phi + y)\, g \\ g(T,y) = 1 \end{cases}$$

$$\mathcal{L}^y = \beta\left(-\frac{\rho\sigma}{\beta} B_t - y\right)\partial_y + \frac{1}{2}\eta^2 \partial_{yy}$$

it's all affine again! $g(t,y) = e^{C(t) - D(t)\, y}$

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