- Collection of sources of uncertainty $\underbrace{(X_t^1, X_t^2, \dots, X_t^n)}_{\substack{X_t \text{ - column vector} \\ (n \times 1)}}$ $0 \le t \le T$

$$dX_t = \mu_t^X \, dt + \sigma_t^X \, dW_t$$

$(\sigma_t^X)_{ij} = \sigma_{ij}^X(t, X_t^1, \dots, X_t^n)$ $(n \times n)$

$(n \times 1)$ $\begin{pmatrix} \mu_1^X(t, X_t^1, \dots, X_t^n) \\ \mu_2^X(t, X_t^1, \dots, X_t^n) \\ \vdots \\ \mu_n^X(t, X_t^1, \dots, X_t^n) \end{pmatrix}$ $\quad \begin{pmatrix} W_t^1 \\ W_t^2 \\ \vdots \\ W_t^n \end{pmatrix}$ independent $\mathbb{P}$- B.mtn. $(n \times 1)$

$$Y_i = \sum_j \sigma_{ij} Z_j \,, \quad Z_j \sim N(0,1) \text{ iid.}$$

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} ; \Omega \right)$$

$$\Omega_{ij} = \mathbb{C}[Y_i, Y_j] = \mathbb{E}[Y_i Y_j]$$

$$= \mathbb{E}\left[\sum_{lk} \sigma_{il} Z_l \, \sigma_{jk} Z_k \right] = \sum_{lk} \sigma_{il} \sigma_{jk} \, \mathbb{E}[Z_l Z_k]$$

$$\delta_{lk} = \begin{cases} 1, & l = k \\ 0 & l \neq k \end{cases}$$

$$= \sum_k \sigma_{ik} \sigma_{jk} = (\sigma \sigma^T)_{ij}$$

Cholesky Decomposition of Ω.

- Bank account $(B_t)_{0 \le t \le T}$

$$dB_t = r_t B_t \, dt$$

$$r_t = r(t, X_t^1, \dots, X_t^n)$$

- traded risky assets $\underbrace{(f_t^1, \dots, f_t^n)}_{f_t}$ $0 \le t \le T$

$$\underset{n \times 1}{df_t} = \underset{n \times 1}{\mu_t^f} \, dt + \underset{n \times n}{\sigma_t^f} \, \underset{n \times 1}{dW_t}$$

$$\mu_t^{f,k} = \mu^{f,k}(t, X_t^1, \dots, X_t^n)$$

$$\sigma_t^{f,ij} = \sigma_{ij}^f(t, X_t^1, \dots, X_t^n)$$

- claim $(g_t)_{0 \le t \le T}$ which pays $G_1(X_T) \dots G_n(X_T)$ at T.

× claim (g_t) o/ten which pays $G_1(X_T) \,..\, G_n(X_T)$ at T.

$$\underset{n\times1}{dg_t} = \underset{n\times1}{\mu^g_t}\, dt + \underset{n\times n}{\sigma^g_t}\, \underset{n\times1}{dW_t}$$

dynamic - hedging:

$$(\underset{n\times1}{\alpha_t}, \underset{1}{\beta_t}, \underset{1}{-1}) \quad \text{in} \quad (\underset{n\times1}{f_t}, \underset{1}{B_t}, \underset{1}{g_t})$$

$$V_t = \alpha_t' f_t + \beta_t B_t - g_t$$

$$V_0 = 0$$

self-financing

$$dV_t = \alpha_t' df_t + \beta_t dB_t - dg_t$$

$$= \alpha_t' (\mu^f_t dt + \sigma^f_t dW_t) + r\beta_t B_t dt$$

$$- (\mu^g_t dt + \sigma^g_t dW_t)$$

$$= \underbrace{(\alpha_t' \mu^f_t + r\beta_t B_t - \mu^g_t)}_{A_t} dt + \underbrace{(\alpha_t' \sigma^f_t - \sigma^g_t)}_{\text{to locally remove risk } = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}} dW_t$$

$$\Rightarrow \quad \boxed{\alpha_t' = \sigma^g_t (\sigma^f_t)^{-1}}$$

× $\Rightarrow$ $dV_t = A_t\, dt$, $A_t \in \mathcal{F}_t$ $\Rightarrow$ $A_t = 0$ to avoid arbitrage.

$$\Rightarrow \quad dV_t = 0 \quad \Rightarrow \quad V_t = 0 \quad \Rightarrow \quad \beta_t B_t = g_t - \alpha_t' f_t$$

× $\alpha_t' \mu^f_t + r_t(g_t - \alpha_t' f_t) - \mu^g_t = 0$

$$\Rightarrow \quad \alpha_t' (\mu^f_t - r_t f_t) = \mu^g_t - r_t g_t$$

$$\Rightarrow \quad \sigma^g_t (\sigma^f_t)^{-1} (\mu^f_t - r_t f_t) = \mu^g_t - r_t g_t$$

$\Rightarrow$

$$\boxed{(\sigma_t^f)^{-1}(\mu_t^f - r_t f_t) = (\sigma_t^g)^{-1}(\mu_t^g - r_t g_t)}$$

$= \underset{(n\times 1)}{\lambda_t}$ market price of risk.

so $(\sigma_t^g)^{-1}(\mu_t^g - r_t g_t) = \lambda_t$

$\Rightarrow \quad \mu_t^g - r_t g_t = \sigma_t^g \lambda_t$

$\Rightarrow \quad \mu_t^g - \sigma_t^g \lambda_t = r_t g_t \longleftarrow$

recall that,

$$dg_t^i = \underbrace{(\partial_t g_t^i + \mathcal{L}_t^X g_t^i)}_{\mu_t^g} dt + \sum_{jk} \partial_{x_j} g_t^i \cdot \sigma^X_{t\,jk}\, dW_t^k$$

$$\sigma^g_{t\,ik} = \sum_j \partial_{x_j} g_t^i \, \sigma^X_{t\,jk}$$

$$\Rightarrow \quad \partial_t g_t^i + \mathcal{L}_t^X g_t^i - \sum_k \sigma^g_{t\,ik} \lambda_{tk} = r_t g_t^i$$

$$\Rightarrow \quad \partial_t g_t^i + \mathcal{L}_t^X g_t^i - \sum_{kj} \partial_{x_j} g_t^i \, \sigma^X_{t\,jk} \lambda_{tk} = r_t g_t^i$$

must hold $\forall (t, X_t) \Rightarrow$ holds for the function g

$$\Rightarrow \quad \boxed{\begin{aligned} &\partial_t g^i + \sum_j \partial_{x_j} g^i \Big(\mu^X_j - \sum_k \sigma^X_{jk} \lambda_k \Big) \\ &\quad + \frac{1}{2} \sum_{jk} \partial_{x_j x_k} g^i \, (\sigma^X \sigma^{X\prime})_{jk} = r_t g^i \end{aligned}}$$

Multi-variate generalized Pricing PDE.

Feynman-Kac:

$$g(t,x) = \mathbb{E}^{\mathbb{P}^*}\left[e^{-\int_t^T r_s\, ds} G(X_T) \,\middle|\, X_t = x \right]$$

$$dX_t = (\mu_t^X - \sigma_t^X \lambda_t)\, dt + \sigma_t^X\, dW_t^*$$

$$d X_t = (\mu_t^X - \sigma_t^X \lambda_t)\, dt + \sigma_t^X \, dW_t^*$$

$\hookrightarrow \mathbb{P}^* -$ B. mtn.

If all X were traded, then, $\mu_t^X - \sigma_t^X \lambda_t = r_t X_t$

W_t — B. mtn.

$$f_t = f(t, W_t)$$

$$df_t \text{ "=" } f(t+dt, W_{t+dt}) - f(t, W_t)$$

$$= f(t+dt, W_t + dW_t) - f(t, W_t)$$

$$= \partial_t f(t, W_t)\, dt + \partial_w f(t, W_t)\, dW_t + \tfrac{1}{2} \partial_{ww} f(t, W_t) (dW_t)^2 + \cdots$$

dt

$$= \partial_t f_t\, dt + \partial_w f_t\, dW_t + \tfrac{1}{2} \partial_{ww} f_t\, dt + \cdots$$

$W_t^1, \ldots, W_t^n$ - independent B. mtn.

$$f_t = f(t, W_t^1, \ldots, W_t^n)$$

$$df_t \text{ "=" } f(t+dt, W_t^1 + dW_t^1, \ldots, W_t^n + dW_t^n) - f_t$$

$$= \partial_t f\, dt + \partial_{w^1} f_t\, dW_t^1 + \partial_{w^2} f_t\, dW_t^2 + \cdots + \partial_{w^n} f_t\, dW_t^n + \tfrac{1}{2} \sum_{ij} \partial_{w^i w^j} f_t \cdot dW_t^i\, dW_t^j + \cdots$$

$\hookrightarrow \delta^{ij}\, dt$

$$= \left[\partial_t f_t + \tfrac{1}{2} \sum_{ij} \delta_{ij}\, \partial_{w^i w^j} f_t\right] dt + \sum_i \partial_{w^i} f_t\, dW_t^i + \cdots$$

$W_t^1, \ldots, W_t^n$

$X_t^1, \ldots, X_t^n$, $\quad \underset{n\times n}{dX_t} = \underset{n\times 1}{\mu_t^X}\, dt + \underset{n\times n}{\sigma_t^X}\, \underset{n\times 1}{dW_t}$

$$f_t = f(t, X_t^1, \ldots, X_t^n)$$

$$df_t \text{ "=" } \partial_t f_t\, dt + \sum_i \partial_{x^i} f_t\, dX_t^i + \tfrac{1}{2} \sum_{ij} \partial_{x^i x^j} f_t\, (dX_t^i\, dX_t^j) + \cdots$$

$$dX_t^i \, dX_t^j = (\ )\, dt^2 + (\ )\, dt\, dW_t$$
$$+ \sum_k \sigma^X_{t\,ik}\, dW_t^k \sum_\ell \sigma^X_{t\,j\ell}\, dW_t^\ell$$
$$= \sum_{k,\ell} \sigma^X_{t\,ik}\, \sigma^X_{t\,j\ell} \cdot \underbrace{dW_t^k\, dW_t^\ell}_{\hookrightarrow\ \delta^{k\ell}\, dt}$$
$$= \sum_k \sigma^X_{t\,ik}\, \sigma^X_{t\,jk}\, dt = (\underbrace{\sigma^X_t\, \sigma^{X'}_t}_{\Omega^X_t})_{ij}\, dt$$

$$\boxed{\begin{aligned} df_t &= \Big(\partial_t f_t + \underbrace{\sum_i \mu_t^{X,i}\, \partial_{x_i} f_t + \tfrac{1}{2} \sum_{ij} (\sigma^X_t \sigma^{X'}_t)_{ij}\, \partial_{x_i x_j} f_t}_{\mathcal{L}^X_t f_t} \Big)\, dt \\ &\quad + \sum_{i,j} \partial_{x_i} f_t\, \sigma^X_{t\,ij}\, dW_t^j \end{aligned}}$$

$$\mathcal{L}^X_t h(t,x) = \lim_{\Delta t \downarrow 0} \frac{\mathbb{E}[\, h(t, X_{t+\Delta t}) \mid X_t = x] - h(t,x)}{\Delta t}$$

$\hookrightarrow$ infinitesimal generator of X.

volatility

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dW_t^*, \qquad r=0.$$

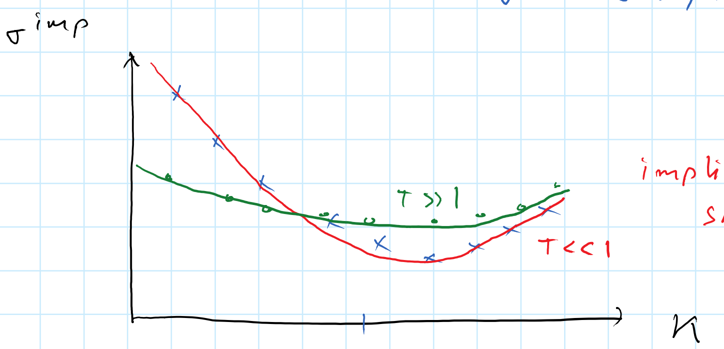
Call option $C(t, S; T, K, \sigma) = \cdots$

$C^{Mkt}(0, 100; 1\,yr, 200) = \#$

$\Rightarrow$ σ^{imp} : implied volatility

$$\boxed{C^{BS}(t, S; T, K, \sigma^{imp}) = C^{Mkt}(t, S; T, K)}$$

$\sigma^{imp}(T, K)$



implied volatility smile / smirk

Heston Model:

$$\frac{dF_t}{F_t} = \sqrt{v_t}\,dW_t^*$$

$\mathbb{P}^*$ - B.mtn. $[W, B]_t = \rho t$

level of m-r.

$$dv_t = \kappa(\theta - v_t)\,dt + \eta\sqrt{v_t}\,dB_t^*$$

rate of mean-reversion

vol-vol

Feller process $2\kappa\theta > \eta$

$$dW_t^* = \lambda_t^F\,dt + dW_t$$

$$dB_t^* = \lambda_t^v\,dt + dB_t$$

$a v_t^{-1/2} + b v_t^{1/2}$

$$dv_t = \kappa(\theta - v_t)\,dt + \eta\sqrt{v_t}\left(\left(a v_t^{-1/2} + b v_t^{1/2}\right)dt + dB_t\right)$$

$$= \left((\kappa\theta + a\eta) - (\kappa - b\eta)v_t\right)dt + \eta\sqrt{v_t}\,dB_t$$

$$= \underbrace{(\kappa - b\eta)}_{\kappa^{P}} \left(\underbrace{\left(\frac{\kappa\theta + a\eta}{\kappa - b\eta}\right)}_{\theta^{P}} - v_t \right) dt + \eta \sqrt{v_t}\, dB_t$$

① $g(t,F,v) = \mathbb{E}^{P^\alpha}\left[(F_T - K)_+ \mid F_t = F,\ v_t = v \right]$

② Pricing PDE: $(\partial_t + \mathcal{L}^{F,v})\, g = 0$

$$\mathcal{L} g = 0 \cdot \partial_F g + \kappa(\theta - v)\, \partial_v g$$
$$+ \frac{1}{2}\left(F^2 v\, \partial_{FF} g + \eta^2 v\, \partial_{vv} g + 2\rho \sqrt{v} \cdot \eta\sqrt{v} \cdot \partial_{vF}\, g \right)$$

How to simulate (F_t, v_t) ...

$$x_t = \log F_t \Rightarrow \quad dx_t = -\frac{1}{2} v_t\, dt + \sqrt{v_t}\, dW_t^*$$

$$d(\log F_t) = \left(0 + 0 \cdot \left(\frac{1}{F_t}\right) + \frac{1}{2}\left(\cancel{F_t}\sqrt{v_t}\right)^2 \cdot \left(-\frac{1}{\cancel{F_t^2}}\right)\right) dt$$
$$+ \cancel{F_t}\sqrt{v_t} \cdot \left(\frac{1}{\cancel{F_t}}\right) dW_t^*$$

$$x_{t_n} = x_{t_{n-1}} - \frac{1}{2} v_{t_{n-1}}^{+}\, \Delta t + \left(v_{t_{n-1}}^{+}\right)^{1/2} Z_n^1 \sqrt{\Delta t}$$

$$v_{t_n} = v_{t_{n-1}} + \kappa(\theta - v_{t_{n-1}}^{+})\, \Delta t + \eta\left(v_{t_{n-1}}^{+}\right)^{1/2} \left(\rho Z_n^1 + \sqrt{1-\rho^2}\, Z_n^2\right)\sqrt{\Delta t}$$

$$Z_n^1,\ Z_n^2 \sim \mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}; \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$$

$$v^+ = \max(v, 0)$$

$$\frac{dF_t}{F_t} = \gamma \sqrt{v_t}\, dW_t^a$$

$$dv_t = \kappa(\theta - v_t)\, dt + \alpha \sqrt{v_t}\, dB_t^\kappa$$