- underlying source of uncertainty $(X_t)_{0\le t\le T}$ it is an Ito process and satisfies the SDE:

$$dX_t = \mu^X(t,X_t)\,dt + \sigma^X(t,X_t)\,dW_t$$

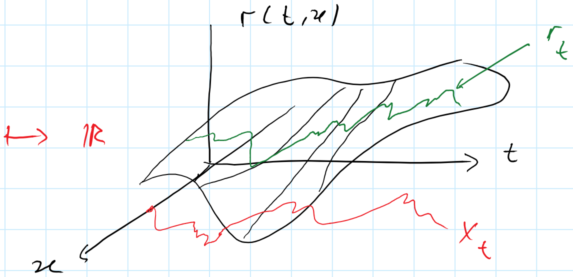
$\mu^X(t,x):\mathbb{R}_+\times\mathbb{R}\mapsto\mathbb{R}$ $\sigma^X(t,x):\mathbb{R}^+\times\mathbb{R}\mapsto\mathbb{R}_+$

X_t is not the price of a traded asset

- Bank account $(B_t)_{0\le t\le T}$ is driven by X through the short rate process $(r_t)_{0\le t\le T}$

$$r_t = r(t,X_t)$$

$r:\mathbb{R}_+\times\mathbb{R}\mapsto\mathbb{R}$



B satisfies the SDE: $dB_t = r_t B_t\,dt$

- Traded risky asset $(F_t)_{0\le t\le T}$ and we assume it is Ito process & that F satisfies SDE

$$dF_t = F_t\,\mu^f(t,X_t)\,dt + F_t\,\sigma^f(t,X_t)\,dW_t$$

instantaneous return of asset or drift. volatility

- $\mu^X_t = \mu^X(t,X_t)$, $\sigma^X_t = \sigma^X(t,X_t)$

$\mu^f_t = \mu^f(t,X_t)$, $\sigma^f_t = \sigma^f(t,X_t)$

- Throw in a third asset $(g_t)_{0\le t\le T}$ and pays $G(X_T)$ at maturity T.

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What properties must g have to avoid arbitrage?

- set up a self-financing strategy $(\alpha_t, \beta_t)_{0\le t\le T}$ and short 1 of g. Let $(V_t)_{0\le t\le T}$ be the value of the strategy.

α_t: # of f $\quad$ β_t: # of B

$$V_t = \alpha_t f_t + \beta_t B_t - g_t$$

- specifically set $V_0 = 0$.

- $$\begin{aligned} dV_t &= d(\alpha_t f_t) + d(\beta_t B_t) - dg_t \\ &= d\alpha_t\, f_t + \alpha_t\, df_t + d[\alpha, f]_t \\ &\quad + d\beta_t\, B_t + \beta_t\, dB_t + d[\beta, B]_t \quad - dg_t \end{aligned}$$

self-financing condition

$$= \boxed{\alpha_t}\, df_t + \boxed{\beta_t}\, dB_t - \boxed{1}\, dg_t$$

let's assume that $g_t = g(t, X_t)$ and $g \in C^{1,2}$ then

what is this function g.

$$dg_t = g_t\, \mu^g(t, X_t)\, dt + g_t\, \sigma^g(t, X_t)\, dW_t$$

$$\Rightarrow dV_t = \alpha_t\,(f_t\, \mu_t^f\, dt + f_t\, \sigma_t^f\, dW_t) + \beta_t\, r_t\, B_t\, dt$$
$$- (g_t\, \mu_t^g\, dt + g_t\, \sigma_t^g\, dW_t)$$

$$= \underbrace{(\alpha_t f_t\, \mu_t^f - g_t\, \mu_t^g + \beta_t\, r_t\, B_t)}_{A_t} dt$$
$$+ (\alpha_t f_t\, \sigma_t^f - g_t\, \sigma_t^g)\, dW_t$$

locally remove risk:

$$\boxed{\alpha_t = \frac{g_t\, \sigma_t^g}{f_t\, \sigma_t^f}}$$

now: $dV_t = A_t\, dt + A_t\, t\, F_t$

$\Rightarrow$ to avoid arbitrage, we must have $A_t = 0 \ \forall\, (t, X_t)$

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$$\Rightarrow \quad dV_t = 0 \quad \forall\, t$$

$$\Rightarrow \quad V_t = 0 \quad \forall\, t$$

$$\Rightarrow \quad \alpha_t f_t + \beta_t B_t - g_t = 0$$

$$\Rightarrow \quad \boxed{\beta_t B_t = g_t - \alpha_t f_t}$$

$$\mathcal{A}_t = 0 \iff \alpha_t f_t \mu^f_t + r_t \beta B_t = \mu_t^g g_t$$

$$\iff \mu_t^f \frac{\sigma_t^g}{\sigma_t^f} \cancel{g_t} + r_t \left(\cancel{g_t} - \frac{\sigma_t^g}{\sigma_t^f} \cancel{g_t} \right) = \mu_t^g \cancel{g_t}$$

$$\iff \boxed{\frac{\mu_t^f - r_t}{\sigma_t^f} = \frac{\mu_t^g - r_t}{\sigma_t^g}} = \lambda_t = \lambda(t, X_t)$$

"Sharpe ratio"

has to be the same (λ_t) $0 \le t \le T$ ∀ traded assets!

"Market price of risk"

$$\frac{\mu_t^g - r_t}{\sigma_t^g} = \lambda_t \iff \mu_t^g - \lambda_t \sigma_t^g = r_t$$

recall Itô's lemma says that $g_t = g(t, X_t)$ satisfies the SDE

$$dg_t = \Big[\partial_t g(t,X_t) + \mu_t^X(t,X_t)\, \partial_x g(t,X_t) + \tfrac{1}{2}\big(\sigma_t^X(t,X_t)\big)^2 \partial_{xx} g(t,X_t) \Big] dt + \sigma_t^X(t,X_t)\, \partial_x g(t,X_t)\, dW_t$$

$$\partial_t g_t := \partial_t g(t, X_t)$$
$$\partial_x g_t := \partial_x g(t, X_t)$$

$$dg_t = \overbrace{\left(\partial_t g_t + \mu_t^X \partial_x g_t + \tfrac{1}{2}(\sigma_t^X)^2 \partial_{xx} g_t \right)}^{g_t \mu_t^g} dt + \underbrace{\left(\sigma_t^X \partial_x g_t \right)}_{\approx\, g_t \sigma_t^g} dW_t$$

$$\mu_t^g - \lambda_t \sigma_t^g = r_t \quad \Rightarrow \quad \mu_t^g g_t - \lambda_t \sigma_t^g g_t = r_t g_t$$

$$\Rightarrow \quad \partial_t g_t + \mu_t^X \partial_x g_t + \tfrac{1}{2}(\sigma_t^X)^2 \partial_{xx} g_t - \lambda_t \sigma_t^X \partial_x g_t = r_t g_t$$

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$$\Rightarrow \quad \partial_t g_t + \mu_t^X \partial_x g_t + \tfrac{1}{2}(\sigma_t^X)^2 \partial_{xx} g_t - \lambda_t \sigma_t^X \partial_x g_t = r_t g_t$$

$$\Rightarrow \quad \partial_t g_t + (\mu_t^X - \lambda_t \sigma_t^X) \partial_x g_t + \tfrac{1}{2}(\sigma_t^X)^2 \partial_{xx} g_t = r_t g_t$$

holds $\forall$ $(t,x) \in [0,T] \times \mathbb{R}$ and so the function g satisfies the PDE:

$$\boxed{\begin{aligned} &\partial_t g(t,x) + (\mu^X(t,x) - \lambda(t,x)\sigma^X(t,x)) \partial_x g(t,x) \\ &\qquad + \tfrac{1}{2}(\sigma^X(t,x))^2 \partial_{xx} g(t,x) = r(t,x)\, g(t,x) \\ &\text{terminal condition} \quad g(T,x) = G(x) \end{aligned}}$$

generalized Black-Scholes PDE.

If X_t is traded, e.g. by choosing $f_t = X_t$

$$\begin{aligned} \Rightarrow \quad df_t &= \mu_t^X\, dt + \sigma_t^X\, dW_t \\ &= f_t \mu_t^f\, dt + f_t \sigma_t^f\, dW_t \\ &= X_t \mu_t^f\, dt + X_t \sigma_t^f\, dW_t \end{aligned}$$

$$\Rightarrow \quad \underline{\mu_t^f} = \underline{\mu_t^X / X_t}, \quad \sigma_t^f = \underline{\sigma_t^X / X_t}$$

$$\boxed{\frac{\mu_t^f - r_t}{\sigma_t^f} = \lambda_t} \quad \Rightarrow \quad \mu_t^f - \lambda_t \sigma_t^f = r_t$$

$$\Rightarrow \quad \mu_t^X - \lambda_t \sigma_t^X = r_t X_t$$

when X is traded we have:

$$\left\{ \begin{aligned} &\partial_t g + r x \partial_x g + \tfrac{1}{2}(\underset{\sigma^X(t,x)}{\sigma^X})^2 \partial_{xx} g = \overset{r(t,x)}{r} g \\ &g(T,x) = G(x) \end{aligned} \right.$$

try repeating with X traded from the start.

suppose that X is traded and $\frac{dX_t}{X_t} = \mu\, dt + \sigma\, dW_t$

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suppose that X is traded and $\frac{dX_t}{X_t} = \mu\, dt + \sigma\, dW_t$

(i.e. $dX_t = \underbrace{X_t\, \mu}_{\mu^X}\, dt + \underbrace{\sigma\, X_t}_{\sigma^X} dW_t$ (Black-Scholes model)

$$\Rightarrow \quad \begin{cases} \partial_t g + r x\, \partial_x g + \frac{1}{2} \sigma^2 x^2\, \partial_{xx} g = r g, \\ g(T, x) = G(x) \end{cases}$$

Black-Scholes PDE.

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$$\begin{cases} \partial_t g + r x \, \partial_x g + \frac{1}{2}\sigma^2 x^2 \, \partial_{xx} g = r g \\ g(T, x) = 1 . \end{cases}$$

$r(t,x) = r(t)$ only a fn. of time

this is a bond with deterministic discounting, so expect that

$g(t,x) = g(t)$ (only a fn of t)

$\partial_t g(t) = r(t) \, g(t), \qquad g(T) = 1$

$\Rightarrow \dfrac{\partial_t g(t)}{g(t)} = r(t) \Rightarrow \partial_t \log g(t) = r(t)$

$\Rightarrow \int_t^T \partial_t \log g(u) \, du = \int_t^T r(u) \, du$

$\Rightarrow \log \overset{1}{g(T)} - \log g(t) = \int_t^T r(u) \, du$

$\Rightarrow \boxed{g(t) = e^{-\int_t^T r(u) \, du}}$

$$\begin{cases} \partial_t g + r x \, \partial_x g + \frac{1}{2}\sigma^2 x^2 \, \partial_{xx} g = r g \\ g(T, x) = x \end{cases}$$

this is just a claim that pays X_T at T, so it is no different from the asset itself.

$g(t,x) \overset{?}{=} x$

$\text{l.h.s.} = 0 + r x \cdot 1 + \frac{1}{2}\sigma^2 x^2 \cdot 0$ ✓

$\text{r.h.s.} = r x$

terminal condition $g(T,x) = x$ ✓

$$\begin{cases} \partial_t g + r x \, \partial_x g + \frac{1}{2}\sigma^2 x^2 \, \partial_{xx} g = r g \\ g(T, x) = x^2 \end{cases}$$

$2(r - \frac{1}{2}\sigma^2) T + 2\sigma\sqrt{T} Z$

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$$g(T,x) = x^2$$

$$\mathbb{E}_0^{\mathbb{Q}}[\ X_T^2\ ] = \mathbb{E}_0^{\mathbb{Q}}[X_0^2 e^{2(r-\frac{1}{2}\sigma^2)T + 2\sigma\sqrt{T}\,Z}]$$

$$Z \overset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

$$= X_0^2\ \mathbb{E}^{\mathbb{Q}}[\ -\ ]$$

perhaps $g(t,x) = \ell(t)\, x^2$

$g(T,x) = x^2 \Rightarrow \ell(T) = 1$

$$\Rightarrow x^2\, \partial_t \ell + r x \,.\, 2x \,.\, \ell + \tfrac{1}{2}\sigma^2 x^2 \,.\, 2\,\ell = r\,\ell\, x^2$$

$$\partial_t \ell + (2r + \sigma^2)\,\ell = r\,\ell$$

$$\partial_t \ell + (r + \sigma^2)\,\ell = 0$$

$$\Rightarrow \quad \ell(t) = e^{+(r+\sigma^2)(T-t)}$$

$$g(t,x) = e^{(r+\sigma^2)(T-t)}\, x^2$$

try $G(x) = x^{\alpha}$

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solving parabolic PDEs.

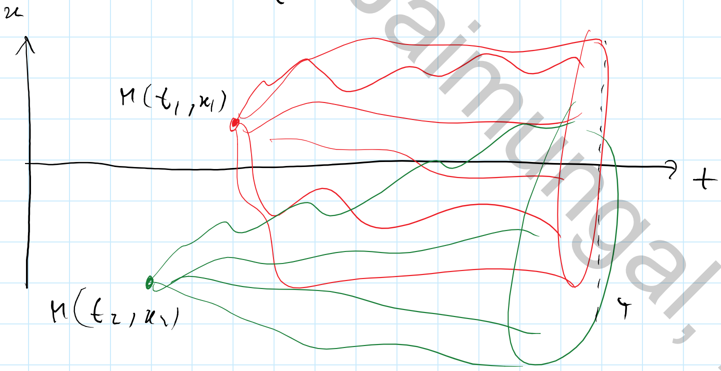
$$\begin{cases} \partial_t h(t,x) + \frac{1}{2}\partial_{xx} h(t,x) = 0, \\ \qquad\qquad h(T,x) = H(x). \end{cases}$$

Feynman-Kac Theorem:

Suppose h satisfies the above, then it admits the following representation:

$$h(t,x) = \mathbb{E}[\, H(X_T) \mid X_t = x]$$

X_t is a Brownian motion.



$g_t \triangleq \mathbb{E}[\, H(X_T) \mid \mathcal{F}_t]$, $\mathcal{F}$ is the natural filtration generated by X.

g_t is in fact a martingale (Doob-martingale)

let $s < t$ and check:

$$\mathbb{E}[\, g_t \mid \mathcal{F}_s] = \mathbb{E}[\, \mathbb{E}[\, H(X_T) \mid \mathcal{F}_t] \mid \mathcal{F}_s]$$

$$= \mathbb{E}[\, H(X_T) \mid \mathcal{F}_s] = g_s$$

law of iterated expectation

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note $g_t = \mathbb{E}[\, H(X_T) \,|\, \mathcal{F}_t]$

$= \mathbb{E}[\, H((X_T - X_t) + X_t) \,|\, \mathcal{F}_t]$

$= \mathbb{E}[\, H((T-t)^{1/2} Z + X_t) \,|\, \mathcal{F}_t]$

$Z \sim \mathcal{N}(0,1) \perp X_t$

$= g(t, X_t) \in C^{1,2}$

for sufficiently well behaved H

Ito for B.M.

$$\Rightarrow \quad dg = \left(\partial_t g(t,X_t) + \tfrac{1}{2}\partial_{xx} g(t,X_t)\right) dt + \partial_x g(t,X_t)\, dW_t$$

$$\Rightarrow \quad \int_t^{t+h} \cdots = \int_t^{t+h} \cdots \qquad (h>0)$$

$$g(t+h, X_{t+h}) - g(t, X_t) = \int_t^{t+h} \left(\partial_t g(u,X_u) + \tfrac{1}{2}\partial_{xx} g(u,X_u)\right) du + \int_t^{t+h} \partial_x g(u,X_u)\, dW_u$$

$\mathbb{E}[\;\cdot\; | \; X_t = x]$

$$\underbrace{\mathbb{E}[\, g_{t+h} - g_t \,|\, X_t = x]}_{\hookrightarrow 0 \text{ by martingale ppty.}} = \mathbb{E}\left[\int_t^{t+h} (\partial_t + \overset{\frac{1}{2}\partial_{xx}}{\mathcal{L}})\, g(u,X_u)\, du \,\Big|\, X_t = x\right] + \underbrace{\mathbb{E}\left[\int_t^{t+h} \partial_x g(u,X_u)\, dW_u \,\Big|\, X_t = x\right]}_{\hookrightarrow 0 \text{ by Ito integrals are 0 mean.}}$$

$$0 = \lim_{h\downarrow 0} \mathbb{E}\left[\frac{1}{h}\int_t^{t+h} (\partial_t + \mathcal{L})\, g(u,X_u)\, du \,\Big|\, X_t = x\right]$$

$$= \mathbb{E}[\, (\partial_t + \mathcal{L})\, g(t, X_t) \,|\, X_t = x]$$

$$= (\partial_t + \mathcal{L})\, g(t,x)$$

i.e. $\quad \partial_t g(t,x) + \frac{1}{2}\partial_{xx} g(t,x) = 0$

and we also have that $g_T = g(T,x) = \mathbb{E}[\, H(X_T) \,|\, X_T = x]$

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$$= H(x)$$

so $g(t,x)$ satisfies the required PDE!

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$$\begin{cases} \partial_t h(t,x) + a(t,x)\, \partial_x h + \frac{1}{2} b^2(t,x)\, \partial_{xx} h(t,x) = c(t,x)\, h(t,x), \\ h(T,x) = H(x). \end{cases}$$

Fixed & given.

Feynman-Kac Theorem:

$$h(t,x) = \mathbb{E}^{\mathbb{P}^*}\left[e^{-\int_t^T c(s,X_s)\, ds}\, H(X_T) \,\middle|\, X_t = x \right]$$

where X_t satisfies the SDE:

$$dX_t = a(t,X_t)\, dt + b(t,X_t)\, dW_t^*$$

W_t^* is a $\mathbb{P}^*$ - B. mtn.

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$$\partial_t g(t,x) + \left(\mu^x(t,x) - \lambda(t,x)\, \sigma^x(t,x) \right) \partial_x g(t,x) + \frac{1}{2}\left(\sigma^x(t,x)\right)^2 \partial_{xx} g(t,x) = r(t,x)\, g(t,x)$$

terminal condition $g(T,x) = G(x)$

$$\Rightarrow g(t,x) = \mathbb{E}^{\mathbb{P}^*}\left[e^{-\int_t^T r(s,X_s)\, ds}\, G(X_T) \,\middle|\, X_t = x \right]$$

& $dX_t = \left(\mu^x(t,X_t) - \lambda(t,X_t)\, \sigma^x(t,X_t) \right) dt + \sigma^x(t,X_t)\, dW_t^*$

& W_t^* is a $\mathbb{P}^*$ - B. mtn.

?? $\rightarrow$ $dW_t^* = \lambda(t,X_t)\, dt + dW_t$

$dX_t = \mu^x(t,X_t)\, dt + \sigma(t,X_t)\, dW_t$ ← the original "$\mathbb{P}$" SDE for X!

Girsanov's Theorem:

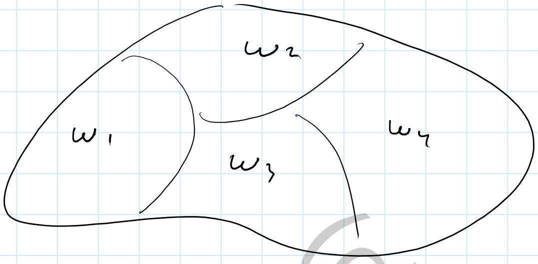
$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \mathcal{E}\left(\int_0^T \lambda_u\, dW_u \right), \quad W_t \text{ is a } \mathbb{P}\text{-B mtn.}$$

└ Dolean-Dade exponential

(stochastic exponential)

then $W^*_t = -\int_0^t \lambda_u \, du + W_t$ is a $\mathbb{P}^*$ – B.motn.

$\mathbb{P} : \Omega \mapsto [0,1] \qquad (\Omega, \mathcal{F}, \mathbb{P})$



$$\mathbb{E}[Y] = \int Y(\omega) \, d\mathbb{P}(\omega)$$
$$= \sum Y(\omega_i) \, \mathbb{P}(\omega_i)$$
$$= \sum \left(Y(\omega_i) \frac{\mathbb{P}(\omega_i)}{\mathbb{P}^*(\omega_i)} \right) \mathbb{P}^*(\omega_i)$$

$\mathbb{P}(\omega_i) = 0$ if $\mathbb{P}^*(\omega_i) = 0)$

$$= \int Y(\omega) \left(\frac{d\mathbb{P}}{d\mathbb{P}^*}\right)(\omega) \, d\mathbb{P}^*(\omega)$$

$\mathbb{P}^*(\omega_i) = 0$ if $\mathbb{P}(\omega_i) = 0$

Radon-Nikodym derivative

$$= \mathbb{E}^{\mathbb{P}^*}\left[Y \cdot \frac{d\mathbb{P}}{d\mathbb{P}^*} \right]$$

$$\mathbb{E}^{\mathbb{P}^*}[Y] = \mathbb{E}^{\mathbb{P}}\left[Y \cdot \frac{d\mathbb{P}^*}{d\mathbb{P}} \right]$$

Radon-Nikodym derivative must

i) $\mathbb{P}\left(\frac{d\mathbb{P}^*}{d\mathbb{P}} > 0 \right) = 1$

ii) $\mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{P}^*}{d\mathbb{P}} \right] = 1$

e.g.:

$$Z \overset{\mathbb{P}}{\sim} \mathcal{N}(0,1)$$

$$\boxed{\eta \triangleq \frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{-\frac{1}{2}\lambda^2 + \lambda Z}}$$

clearly $\eta > 0$ a.s. & $\mathbb{E}^{\mathbb{P}}[\eta] = e^{-\frac{1}{2}\lambda^2} \, \mathbb{E}^{\mathbb{P}}[e^{\lambda Z}]$

$$= e^{-\frac{1}{2}\lambda^2} \cdot e^{\frac{1}{2}\lambda^2} = 1$$

$$\mathbb{E}^{\mathbb{P}^*}[e^{uZ}] = \mathbb{E}^{\mathbb{P}}\left[e^{uZ} \frac{d\mathbb{P}^*}{d\mathbb{P}} \right]$$
$$= \mathbb{E}^{\mathbb{P}}\left[e^{uZ} \cdot e^{-\frac{1}{2}\lambda^2 + \lambda Z} \right]$$
$$= \mathbb{E}^{\mathbb{P}}\left[e^{(u+\lambda)Z} \right] e^{-\frac{1}{2}\lambda^2}$$
$$= e^{\frac{1}{2}(u+\lambda)^2} \, e^{-\frac{1}{2}\lambda^2} = e^{u\lambda + \frac{1}{2}u^2}$$

(exponent $= \frac{1}{2}u^2 + u\lambda + \frac{1}{2}\lambda^2 - \frac{1}{2}\lambda^2$) $\Rightarrow Z \overset{\mathbb{P}^*}{\sim} \mathcal{N}(\lambda, 1)$

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$$= e \qquad e^{-} \qquad = e$$

$$\left(\text{expnent} = \tfrac{1}{2} u^2 + u\lambda + \tfrac{1}{2}\lambda^2 - \tfrac{1}{2}\lambda^2\right) \qquad Z \overset{\mathbb{P}^*}{\sim} \mathcal{N}(\lambda, 1)$$

$$\mathcal{N}(0,1) \overset{\mathbb{P}^*}{\sim} Z^* \triangleq Z - \lambda$$

$$\mathcal{E}\left(\int_0^t \lambda_u \, dW_u\right) \triangleq e^{-\frac{1}{2}\int_0^t \lambda_u^2 \, du + \int_0^t \lambda_u \, dW_u}$$

$$\left(e^{-\frac{1}{2}\lambda^2 + \lambda Z}\right) \xleftarrow{d}$$

$\lambda = \text{const}$,
$t = 1$

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recall that to locally remove risk:

$$\alpha_t = \frac{g_t\,\sigma_t^g}{f_t\,\sigma_t^F}$$

if X_t is traded and so $f_t = X_t \Rightarrow$

$$\alpha_t = \frac{g_t\,\sigma_t^g}{\sigma_t^X}\,,\quad g_t\sigma_t^g = \partial_x g(t, X_t)\cdot\sigma_t^X$$

from Ito's lemma

$$\Rightarrow \boxed{\alpha_t = \partial_x g(t, X_t)}$$

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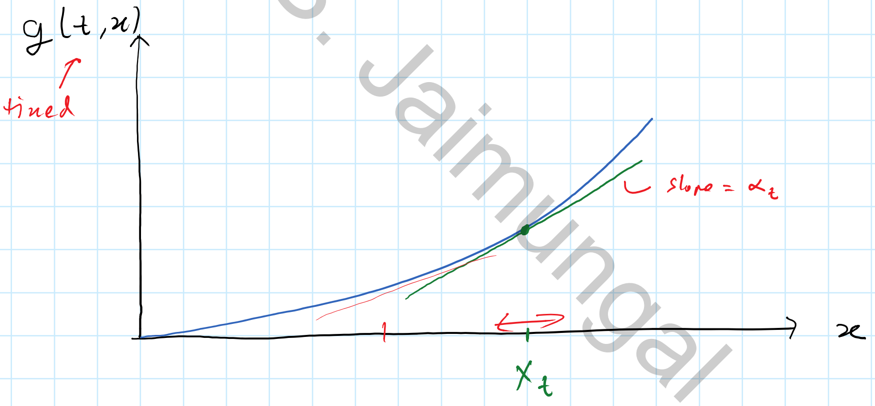

how to use the "α" to hedge option via trading?

1) sell option get $g_0 = g(0, X_0)$

buy α_0 of X, cost $\alpha_0 X_0$

bank account we have $B_0 = g_0 - \alpha_0 X_0$

at t_1: α_0 of X now worth $\alpha_0 X_{t_1}$

bank account " has $B_{t_1^-} = (g_0 - \alpha_0 X_0)\,e^{r\Delta t}$

rebalance $\alpha_0 \rightarrow \alpha_{t_1}$ of X

profit / loss $= (\alpha_0 - \alpha_{t_1})\,X_{t_1}$

$B_1 = (g - \alpha \; X) \, e^{r\Delta t} + (\alpha - \alpha) \, X$

profit / loss $= (\alpha_0 - \alpha_{t_1})\, X_{t_1}$

$$B_{t_1} = (g_0 - \alpha_0 X_0)\, e^{r\Delta t} + (\alpha_0 - \alpha_{t_1})\, X_{t_1}$$

at t_2: α_{t_1} of X now worth $\alpha_{t_1} X_{t_2}$

$$B_{t_2^-} = B_{t_1} e^{r\Delta t}$$

$\alpha_{t_1} \longrightarrow \alpha_{t_2}$ of X, PnL $= (\alpha_{t_1} - \alpha_{t_2})\, X_{t_2}$

$$B_{t_2} = B_{t_1} e^{r\Delta t} + (\alpha_{t_1} - \alpha_{t_2})\, X_{t_2}$$

⋮

at t_k: $\alpha_{t_{k-1}} \longrightarrow \alpha_{t_k}$ of X,

$$\boxed{B_{t_k} = B_{t_{k-1}} e^{r\Delta t} + (\alpha_{t_{k-1}} - \alpha_{t_k})\, X_{t_k}}$$

⋮

at $t_n = T$: $\alpha_{t_{n-1}}$ of X

$B_{t_{n-1}} e^{r\Delta t}$ in Bank

owe $G(X_{t_n})$

$$\boxed{\text{PnL} = \left(\alpha_{t_{n-1}} X_{t_n} + B_{t_{n-1}} e^{r\Delta t}\right) - \left(G(X_{t_n})\right)}$$

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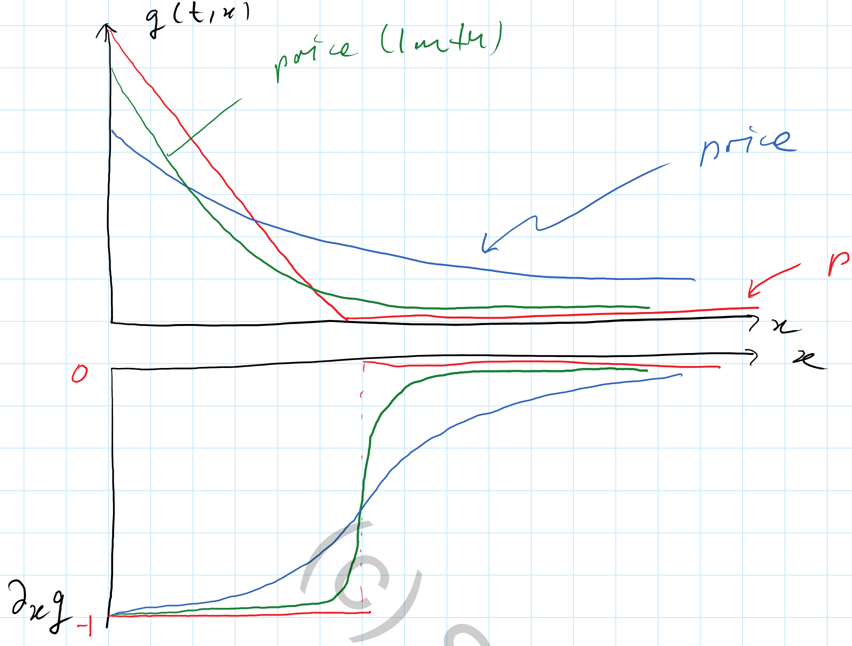
Wednesday, October 8, 2014 7:40 PM



$$g(t,x) = K e^{-r\tau} \Phi(-d_+) - x\, \Phi(-d_-)$$

$$d_\pm = \frac{\log(x/K) + (r \pm \tfrac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$\tau = T - t$$

$$\partial_x g(t,x) = \Phi(-d_-) - 1$$

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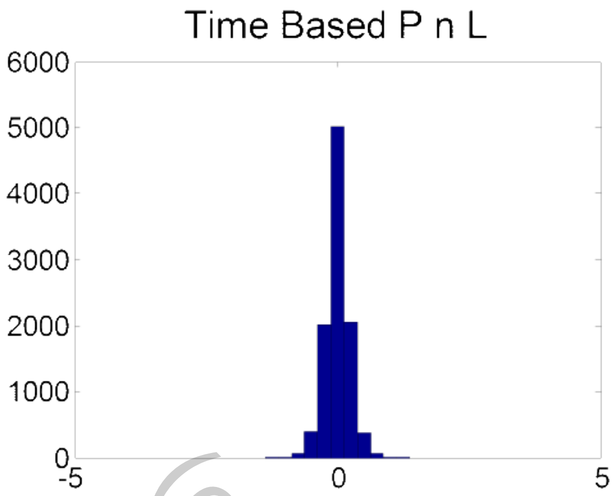

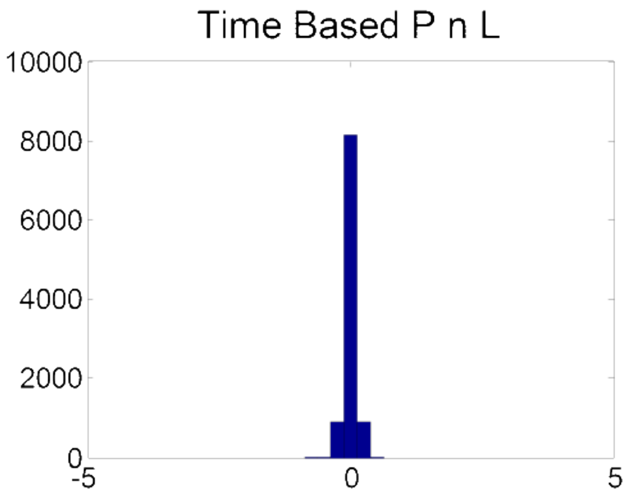

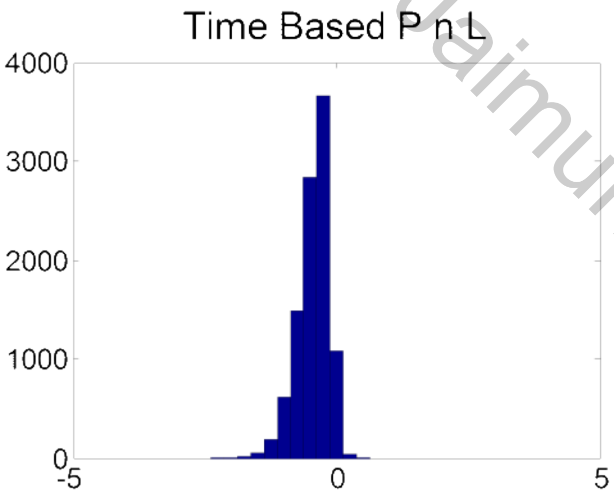

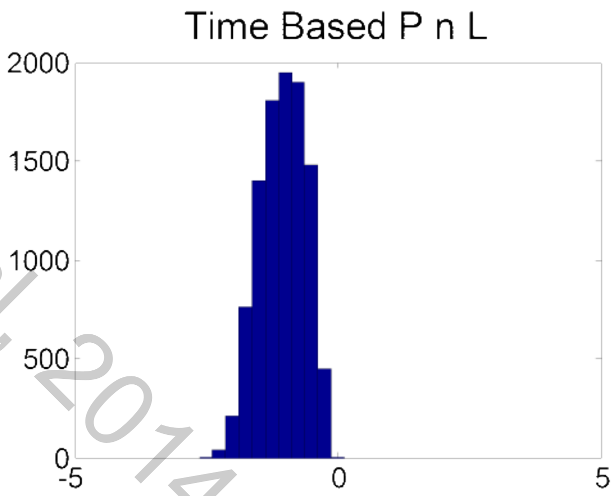

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The "Greeks"

Wednesday, October 8, 2014 8:18 PM

$$g(t, X_t + \Delta X_t) = g(t, X_t) + \Delta X_t \overbrace{\partial_x g(t, X_t)}^{\Delta_t^g \text{ - Delta}}$$

$$+ \frac{1}{2} (\Delta X_t)^2 \underbrace{\partial_{xx} g(t, X_t)}_{\Gamma_t^g \text{ - Gamma}}$$

$$+ \dots$$

$$V_t^h = \alpha_t X_t + \beta_t B_t + \eta_t \overset{\text{hedging option}}{h(t, X_t)}$$

$$\partial_x V_t^h = \alpha_t + \eta_t \partial_x h = \alpha_t + \eta_t \Delta_t^h = \Delta_t^g$$

$$\partial_{xx} V_t^h = \eta_t \partial_{xx} h(t, X_t) = \eta_t \Gamma_t^h = \Gamma_t^g$$

$$\boxed{\eta_t = \frac{\Gamma_t^g}{\Gamma_t^h} \quad \& \quad \alpha_t = \Delta_t^g - \frac{\Gamma_t^g}{\Gamma_t^h} \Delta_t^h}$$

For us to be Delta-Gamma neutral.

$$V_t = \alpha_t X_t + \beta_t B_t + \eta_t h_t - g_t$$

$$\partial_x V_t = 0$$

$$\partial_{xx} V_t = 0$$

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