$$P_0(T) = e^{-y_0(T)\,T}$$

Value a call option on the bond.

$$(P_{T_1}(T_2) - K)_+ \quad \text{is paid at } T_1$$

bond maturity

option maturity

e.g. $T_2 = 3\Delta t$ $\quad K = 0.997$

$T_1 = 2\Delta t$

Wednesday, October 1, 2014 3:23 PM

AR(1) - autoregressive of order 1

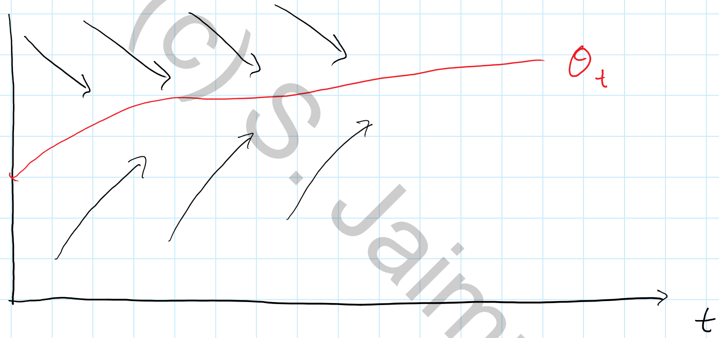
Vasicek - Model

$$r_{t_n} - r_{t_{n-1}} = \kappa\,(\underbrace{\theta_{t_{n-1}} - r_{t_{n-1}}}) \,\Delta t + \sigma\sqrt{\Delta t}\; x_n \,, \qquad t_n = n\,\Delta t,\quad \Delta t = T/n$$

$x_1, x_2, \ldots,$ iid. $\quad \mathbb{E}^{Q}[x_1] = 0$, $\quad \mathbb{V}^{Q}[x_1] = 1$, $\quad \kappa, \sigma > 0$

(e.g. $x_1 \overset{Q}{\sim} N(0,1)$

$x_1 \overset{Q}{\sim} \pm 1$ Bernoulli $\quad Q(x_1 = +1) = 1/2$)



mean-reverting model

$$r_{t_n} = \underbrace{\kappa\,\theta_{t_{n-1}}\Delta t}_{\alpha_{n-1}} + \underbrace{(1-\kappa\Delta t)}_{\beta}\, \underbrace{r_{t_{n-1}}} + \sigma\sqrt{\Delta t}\; x_n$$

$$= \alpha_{n-1} + \beta\left(\alpha_{n-2} + \beta\, r_{t_{n-2}} + \sigma\sqrt{\Delta t}\; x_{n-1}\right) + \sigma\sqrt{\Delta t}\; x_n$$

$$= (\alpha_{n-1} + \beta\,\alpha_{n-2}) + \beta^2 r_{t_{n-2}} + \sigma\sqrt{\Delta t}\,(\beta\, x_{n-1} + x_n)$$

$$= (\alpha_{n-1} + \beta\,\alpha_{n-2}) + \beta^2(\alpha_{n-3} + \beta\, r_{t_{n-3}} + \sigma\sqrt{\Delta t}\; x_{n-2}) + \sigma\sqrt{\Delta t}\,(\beta\, x_{n-1} + x_n)$$

$$= (\alpha_{n-1} + \beta\,\alpha_{n-2} + \beta^2\alpha_{n-3}) + \beta^3 r_{t_{n-3}} + \sigma\sqrt{\Delta t}\,(\beta^2 x_{n-2} + \beta\, x_{n-1} + x_n)$$

$$= \sum_{m=1}^{n} \alpha_{n-m}\,\beta^{m-1} + \beta^n\, r_{t_0} + \underbrace{\sigma\sqrt{\Delta t}\sum_{m=1}^{n} x_m\,\beta^{n-m}}_{X}$$

$t_0 \to 0$

$\kappa\,\theta_{t_{n-m}}\,\Delta t\,\beta^{m-1}$

$\left(1 - \kappa\frac{T}{n}\right)^n \xrightarrow[n\to\infty]{} e^{-\kappa T}$

$$\kappa\,\theta_{t_{n-m}}\,\Delta t\,\beta^{m-1} \qquad \left(1-\kappa\frac{T}{n}\right)^{n} \xrightarrow[n\to\infty]{} e^{-\kappa T} \qquad X$$

$$\sum_{m=0}^{n-1} \alpha_m\,\beta^{n-m-1}$$

$$\kappa\sum_{m=0}^{n-1} \theta_{t_m}\left(1-\frac{\kappa T}{n}\right)^{n-m-1}\Delta t$$

$m = \gamma\, n$ $\qquad \gamma = 0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n-1}{n}$

$$\left(1-\frac{\kappa T}{n}\right)^{(1-\gamma)n-1} \xrightarrow[n\to+\infty]{} e^{-\kappa T(1-\gamma)}$$

$T(1-\gamma) = (T-u)$, u

$$\to \kappa\int_0^T \theta_u\, e^{-\kappa(T-u)}\,du$$

A CLT like result holds for $\underbrace{\sigma\sqrt{\Delta t}\sum_{m=1}^{n} x_m\,\beta^{n-m}}_{X}$ as long as the $\mathbb{E}[X]$ and $\mathbb{V}[X] \xrightarrow[n\to\infty]{}$ converge.

$$\mathbb{E}[X] = 0$$

$$X = \sigma\sqrt{\Delta t}\sum_{m=1}^{n} x_m\,\beta^{n-m} = \sigma\sqrt{\Delta t}\sum_{m=0}^{n-1} x_{n-m}\,\beta^{m}$$

$$\mathbb{V}[X] = \sigma^2\,\Delta t\sum_{m=0}^{n-1} \overset{1}{\mathbb{V}[x_{n-m}]}\,\beta^{2m}$$

$$= \sigma^2 \sum_{m=0}^{n-1} \beta^{2m}\,\Delta t$$

$$\beta^{2m} \hookrightarrow \left(1-\frac{\kappa T}{n}\right)^{2m}, \quad 2m \to \gamma n, \quad \gamma = 0, \tfrac{1}{n}, \tfrac{2}{n}, \dots, \tfrac{n-1}{n} \quad \longrightarrow e^{-2\kappa(T\gamma)}$$

$$\xrightarrow[n\to\infty]{} \int_0^T e^{-2\kappa u}\,du$$

$$\sigma^2\sum_{m=1}^{n} \beta^{2(n-m)}\,\Delta t$$

$$\hookrightarrow \left(1-\frac{\kappa T}{n}\right)^{2(n-m)\to 2n(1-\gamma)} \longrightarrow e^{-2\kappa T(1-\gamma)}$$

$$\xrightarrow[n\to\infty]{} \int_0^T e^{-2\kappa(T-u)}\,du$$

$$\boxed{r_T \overset{d}{=} e^{-\kappa T} r_0 + \kappa\int_0^T \theta_u\, e^{-\kappa(T-u)}\,du + X}$$

$$X \overset{\mathbb{Q}}{\sim} \mathcal{N}\left(0\,;\ \sigma^2\int_0^T e^{-2\kappa u}\,du\right)$$

$$X \overset{\mathbb{Q}}{\sim} \mathcal{N}\left(0 \,;\, \sigma^2 \int_0^T e^{-2\kappa u}\, du \right)$$

$$\hookrightarrow \frac{1 - e^{-2\kappa T}}{2\kappa}$$

$$X \xrightarrow[T \to +\infty]{\mathbb{Q}} \mathcal{N}\left(0 \,;\, \frac{\sigma^2}{2\kappa} \right)$$

$s < t$

$$\boxed{\begin{aligned} r_t &\overset{d}{=} e^{-\kappa(t-s)}\, r_s + \int_s^t \theta_u\, e^{-\kappa(t-u)}\, du + X_{st} \\ X_{st} &\overset{\mathbb{Q}}{\sim} \mathcal{N}\left(0 \,;\, \sigma^2 \int_s^t e^{-2\kappa(t-u)}\, du \right) \end{aligned}}$$

$$P_0(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s\, ds} \right]$$

need to find distribution of $\int_0^T r_s\, ds = \lim_{n\to\infty} \sum_{m=\cancel{0}\,1}^{\cancel{n-1}\,n} r_{\cancel{t_m}\,t_{m-1}} \Delta t$

$$r_{t_m} - r_{t_{m-1}} = \kappa\left(\theta_{t_{m-1}} - r_{t_{m-1}}\right)\Delta t + \sigma\sqrt{\Delta t}\, x_m$$

$$\sum_{m=1}^{n} \left(r_{t_m} - r_{t_{m-1}} \right) = \kappa\, \cancel{\theta\, \underbrace{\Delta t \cdot n}_{T}} - \kappa \sum_{m=1}^{n} r_{t_{m-1}} \Delta t + \sigma\sqrt{\Delta t} \sum_{m=1}^{n} x_m$$

$r_{t_n} - r_0$, $\quad t_n = T$ $\qquad \sum_{m=1}^{n} \theta_{t_{m-1}} \Delta t$

$$\Rightarrow \sum_{m=1}^{n} r_{t_{m-1}} \Delta t = \sum_{m=1}^{n} \theta_{t_{m-1}} \Delta t - \frac{\left(r_T - r_0 \right)}{\kappa} + \frac{\sigma\sqrt{\Delta t}}{\kappa} \sum_{m=1}^{n} x_m$$

$$= \sum_{m=1}^{n} \theta_{t_{m-1}} \Delta t - \left(\sum_{m=1}^{n} \alpha_{n-m} \beta^{m-1} + \left(\beta^n - 1\right) r_{t_0} + \sigma\sqrt{\Delta t} \sum_{m=1}^{n} x_m \beta^{n-m} \right) \frac{1}{\kappa} + \frac{\sigma\sqrt{\Delta t}}{\kappa} \sum_{m=1}^{n} x_m$$

$$\left(\left(1 - \frac{\kappa T}{n} \right)^n - 1 \right) \xrightarrow[n\to\infty]{} e^{-\kappa T} - 1$$

$$\sum_{m=1}^{n} \alpha_{n-m} \beta^{m-1} = \sum_{m=1}^{n} \alpha_{m-1} \beta^{n-m} = \sum_{m=1}^{n} \kappa\, \theta_{t_{m-1}} \left(1 - \frac{\kappa T}{n} \right)^{n-m} \Delta t$$

$$A = \kappa \sum_{m=1}^{n} \theta_{t_{m-1}} \left(1 - \left(1 - \frac{\kappa T}{n} \right)^{n-m} \right) \Delta t$$

$$\xrightarrow[n\to\infty]{} \kappa \int_0^T \theta_u \left(1 - e^{-\kappa(T-u)} \right) du$$

$$C = \frac{\sigma\sqrt{\Delta t}}{\kappa} \sum_{m=1}^{n} x_m \left(1 - \beta^{n-m} \right)$$

$$\mathbb{E}^{\mathbb{Q}}[C] = 0$$

$$\mathbb{V}^{\mathbb{Q}}[C] = \frac{\sigma^2}{\kappa^2} \sum_{m=1}^{n} \left(1 - \beta^{n-m} \right)^2 \Delta t$$

$$\xrightarrow[n\to\infty]{} \frac{\sigma^2}{\kappa^2} \int_0^T \left(1 - e^{-\kappa(T-u)} \right)^2 du$$

$$\boxed{\begin{gathered}\int_0^T r_s\,ds \overset{d}{=} \int_0^T \theta_u\left(1-e^{-\kappa(T-u)}\right)du + \frac{\left(1-e^{-\kappa T}\right)}{\kappa}\,r_0 + Y \\ Y \overset{\mathbb{Q}}{\sim} \mathcal{N}\left(0\,;\ \frac{\sigma^2}{\kappa^2}\int_0^T \left(1-e^{-\kappa(T-u)}\right)^2 du\right)\end{gathered}}$$

$$P_0(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s\,ds}\right] = \exp\left\{-\frac{\left(1-e^{-\kappa T}\right)}{\kappa}r_0 - \int_0^T \theta_u\left(1-e^{-\kappa(T-u)}\right)du + \frac{\sigma^2}{2\kappa^2}\int_0^T\left(1-e^{-\kappa(T-u)}\right)^2 du\right\}$$

$$= \exp\left\{-\int_0^T \underbrace{f_0(u)}_{\text{today's instantaneous forward rates}}\,du\right\}$$

$$= e^{-\underbrace{y_0(T)}_{\text{yields}}\cdot T}$$

because $\boxed{P_0(T) = e^{A_0(T;\theta) - B_0(T)\,r_0}}$ has exponential linear in r_0 form.

This model is said to be affine.

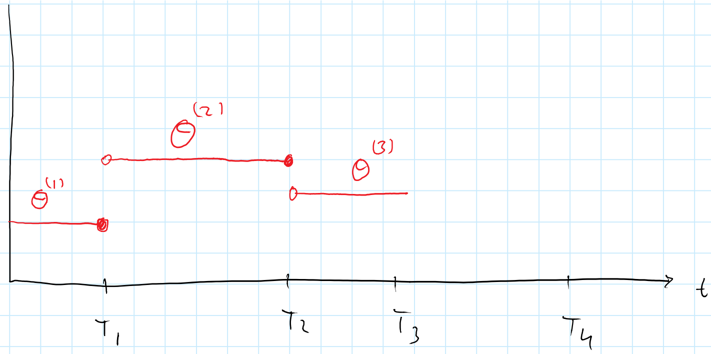
$$\frac{\left(1-e^{-\kappa T}\right)}{\kappa}r_0 + \int_0^T \theta_u\left(1-e^{-\kappa(T-u)}\right)du + \frac{\sigma^2}{2\kappa^2}\int_0^T\left(1-e^{-\kappa(T-u)}\right)^2 du$$

$$= \int_0^T f_0(u)\,du$$

$$\partial_T:\quad \boxed{e^{-\kappa T}r_0} + \theta_T(1-1) + \kappa\boxed{\int_0^T \theta_u\,e^{-\kappa(T-u)}du} + \frac{\sigma^2}{2\kappa^2}\underbrace{\int_0^T 2\left(1-e^{-\kappa(T-u)}\right)\cdot\kappa e^{-\kappa(T-u)}du}_{\ell(T)}$$

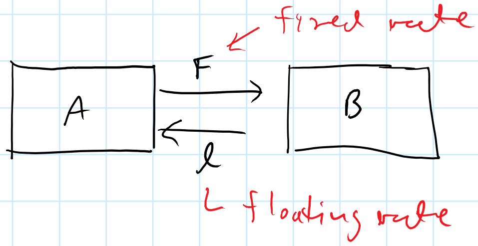
$$= f_0(T)$$

$$\partial_T:\quad \boxed{-\kappa\,e^{-\kappa T}r_0} + \kappa\,\theta_T\cdot 1 - \kappa^2\boxed{\int_0^T \theta_u\,e^{-\kappa(T-u)}du} + \frac{\sigma^2}{2\kappa^2}\,\partial_T\,\ell(T) = \partial_T f_0(T)$$

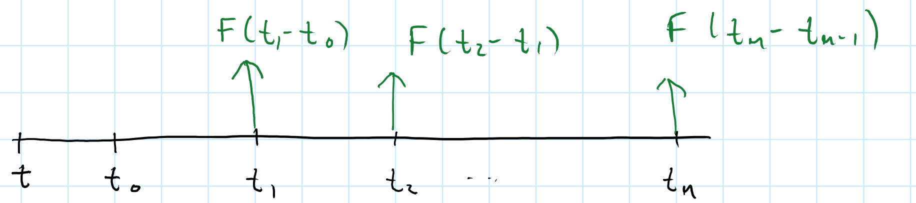


$\theta^{(1)}$
$\theta^{(2)}$
$\theta^{(3)}$
t
T_1
T_2
T_3
T_4

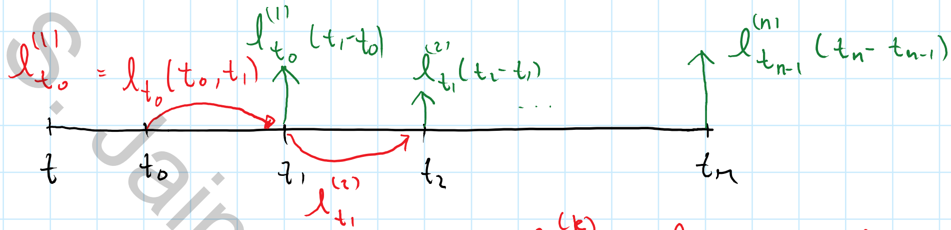
IRS - interest rate swaps



Fixed leg:



Floating leg:

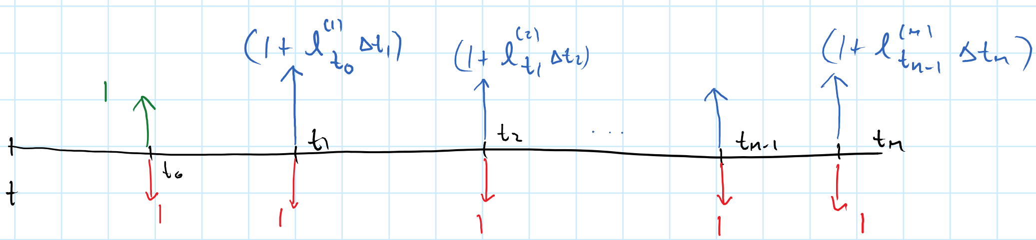


$$\ell_t^{(k)} = \ell_t(t_{k-1}, t_k)$$

$$V_0^{fixed} = \sum_{m=1}^{n} F\,\Delta t_m\, P_0(t_m)$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\sum_{m=1}^{n} F\,\Delta t_m\, e^{-\int_0^{t_m} r_s\,ds}\right]$$

$$V_0^{fl} = \mathbb{E}^{\mathbb{Q}}\left[\sum_{m=1}^{n} \ell_{t_{m-1}}^{(m)}\,\Delta t_m\, e^{-\int_0^{t_m} r_s\,ds}\right]$$



@ t hold t_0-bond, short t_n bond

@ t_0 buy \$1 worth of t_1-bond

@ t_1 buy \$ " " t_2-bond

⋮

$$V^{fl} = P_0(t_0) - P_0(t_n)$$

$$(1+\Delta t_n \ell_{t_{n-1}}^{(n)})^{-1} = P_{1}(t_n) \left(= e^{-y \times (t_n - t_{n-1})} \right)$$

$$(1+\Delta t_k \ell^{(k)}_{t_{k-1}})^{-1} = P_{t_{k-1}}(t_k) \left(= e^{-y \times (t_k - t_{k-1})} \right)$$

$$\Rightarrow \quad \ell^{(k)}_{t_{k-1}} = \frac{1}{\Delta t_k}\left[\frac{1}{P_{t_{k-1}}(t_k)} - 1 \right]$$

LIBOR

(Lond interbank offer rate)

$$\mathbb{E}^{\mathbb{Q}}\left[\ell^{(k)}_{t_{k-1}} \cdot e^{-\int_0^{t_k} r_s\, ds} \right] = V_0^{(k)}$$

$$\frac{V_0^{(k)}}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{\ell^{(k)}_{t_{k-1}}}{B_{t_k}} \right]$$

$$\frac{V_0^{(k)}}{P_0(t_k)} = \mathbb{E}^{\mathbb{Q}^k}\left[\frac{\ell^{(k)}_{t_{k-1}}}{\underbrace{P_{t_k}(t_k)}_{=1}} \right]$$

measure induced by bond - t_k as numeraire asset

$$\ell^{(k)}_{t_{k-1}} \xleftarrow[t \uparrow t_{k-1}]{} \ell^{(k)}_t \triangleq \frac{1}{\Delta t_k}\left[\frac{P_t(t_{k-1})}{P_t(t_k)} - 1 \right]$$

$\ell^{(k)}_t$ is a $\mathbb{Q}^k$ - martingale!

$$\Rightarrow \frac{V_0^{(k)}}{P_0(t_k)} = \mathbb{E}^{\mathbb{Q}^k}\left[\ell^{(k)}_{t=t_{k-1}} \right] \overset{\text{m.c.}}{=} \ell_0^{(k)}$$

$$V_0^{(k)} = P_0(t_k)\, \ell_0^{(k)}$$

$$= \frac{P_0(t_k)}{\Delta t_k}\left[\frac{P_0(t_{k-1})}{P_0(t_k)} - 1 \right]$$

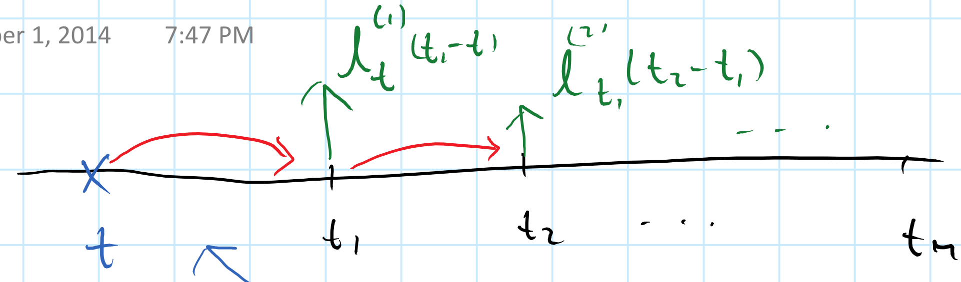
$$\boxed{V_0^{(k)} = \frac{1}{\Delta t_k}\left[P_0(t_{k-1}) - P_0(t_k) \right]}$$

$$V^{f\ell} = \sum_{k=1}^{n} V_0^{(k)} \Delta t_k = \sum_{k=1}^{n} \left(P_0(t_{k-1}) - P_0(t_k) \right)$$

$$\Rightarrow \quad \boxed{V^{fl} = P_0(t_0) - P_0(t_n)}$$

rate F which makes $V_t^{fixed} = V_t^{fl}$ is called the swap-rate :

$$\boxed{S_t = \frac{P_t(t_0) - P_t(t_n)}{\sum_{k=1}^{n} \Delta t_k \, P_t(t_k)}}$$

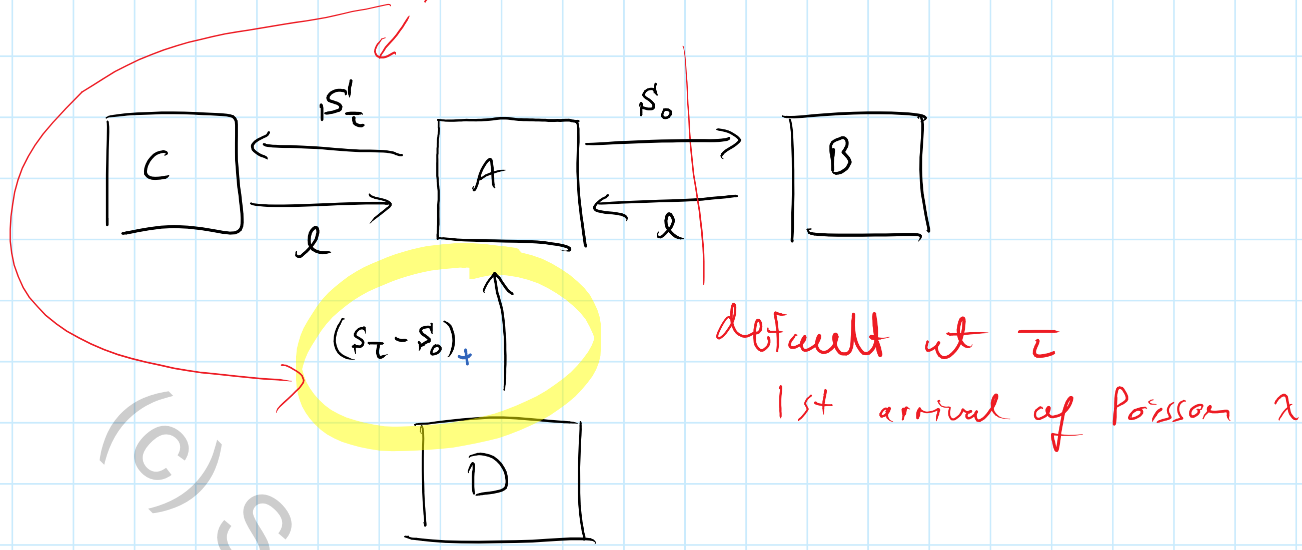


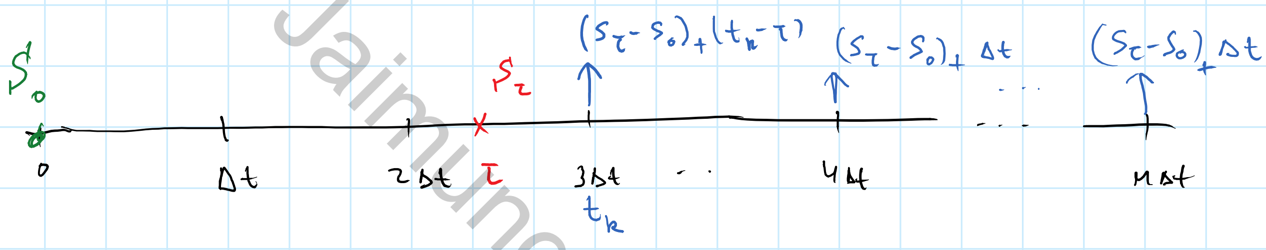
$$S_t = \frac{1 - P_t(t_n)}{\sum_{k:\, t_k > t} \Delta t_k \; P_t(t_k)}$$

$(t_{k_1} - t)$

$(t_{k+1} - t_k)$

CCIRS





$$V_\tau = \mathbb{E}\left[(S_\tau - S_0)_+ \sum_{m=k}^{n} \Delta t_m \, e^{-\int_\tau^{t_m} r_s\, ds} \,\Big|\, \mathcal{F}_\tau \right]$$

$$= (S'_\tau - S'_0)_+ \sum_{m=k}^{n} \Delta t_m \, P_\tau(t_m)$$

annuity

$$S'_\tau = \frac{1 - P_\tau(t_n)}{\sum_{m=k}^{n} \Delta t_m \, P_\tau(t_m)}$$

$$= \left(1 - P_\tau(t_n) - S'_0 \sum_{m=k}^{n} \Delta t_m \, P_\tau(t_m) \right)_+$$

recall that $P_t(T) = e^{A_t(T) - B_t(T)\, r_t}$
(in the Vasicek model)

so we can view $V_\tau = F(\tau, r_\tau)$

$$V_0 = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^\tau r_s\,ds} V_\tau \right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\underbrace{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^\tau r_s\,ds} V_\tau \,\middle|\, \tau \right]} \right]$$

to compute this need joint distribution of $\int_0^u r_s\,ds$ & r_u