FTAP: no arb $\iff \exists\, \mathbb{Q} \sim \mathbb{P}$ s.t. $\forall$ traded assets A, we have

$$\tilde{A}_s = \mathbb{E}^{\mathbb{Q}}[\,\tilde{A}_t \,|\, \mathcal{F}_s\,] \, , \quad s < t$$

$$\tilde{A}_t = A_t / B_t$$

↳ numeraire asset.

numeraire asset: $\mathbb{P}(B_t > 0) = 1 \quad \forall\, t$.

$\tilde{A}_t$ is a $\mathbb{Q}$-martingale.

markets are complete $\iff$ $\mathbb{Q}$ is unique.

$$A_{n+1} = A_n \, e^{\sigma\sqrt{\Delta t}\, x_n} \qquad x_i \ldots \text{ iid } (\pm 1) \text{ Bernoulli.}$$

$$p = \tfrac{1}{2}\left(1 + \tfrac{\gamma}{\sigma}\sqrt{\Delta t}\right)$$

$$\log\left(A_{\underset{\text{↳ } N\Delta T}{T}} / A_0\right) \xrightarrow[N\to\infty]{\mathbb{P}} \mathcal{N}\left(\gamma T \,;\, \underline{\underline{\sigma^2 T}}\right)$$

$$\lim_{N\to\infty} A_T \overset{d}{=} A_0 \, e^{\gamma T + \sigma\sqrt{T} Z} \qquad Z \overset{\mathbb{P}}{\sim} \mathcal{N}(0,1)$$

$$q = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \sim \frac{1}{2}\left(1 + \frac{r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + \ldots$$

$$\log\left(A_T / A_0\right) \xrightarrow[N\to\infty]{\mathbb{Q}} \mathcal{N}\left(\left(r - \tfrac{1}{2}\sigma^2\right)T \,;\, \underline{\underline{\sigma^2 T}}\right)$$

$$\lim_{N\to\infty} A_T \overset{d}{=} A_0 \, e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z^*}$$

$$Z^* \overset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

$$\log(A_T/A_0) \xrightarrow[N\to\infty]{\mathbb{Q}^A} \mathcal{N}\left((r+\tfrac{1}{2}\sigma^2)T;\ \sigma^2 T\right)$$

$$\lim_{N\to\infty} A_T \overset{d}{=} A_0\, e^{(r+\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z^A}$$

$$Z^A \overset{\mathbb{Q}^A}{\sim} N(0,1)$$

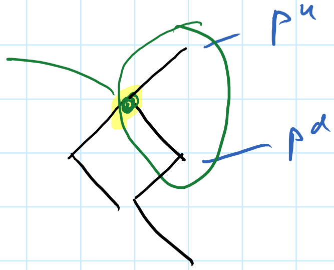
$$\mathbb{E}^{\mathbb{P}}[A_T] = A_0\, e^{\mu T}$$

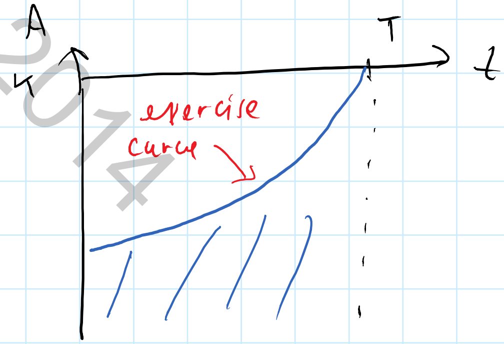
$$\Rightarrow \quad \mu \hat{=} \frac{1}{T} \log \mathbb{E}^{\mathbb{P}}[A_T/A_0] \overset{\substack{N\to\infty \\ CRR}}{=} r + \tfrac{1}{2}\sigma^2$$

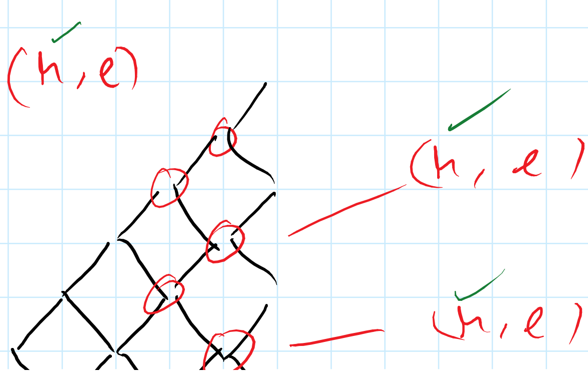
$$\left(\neq \frac{1}{T}\mathbb{E}^{\mathbb{P}}[\log(A_T/A_0)]\right)$$

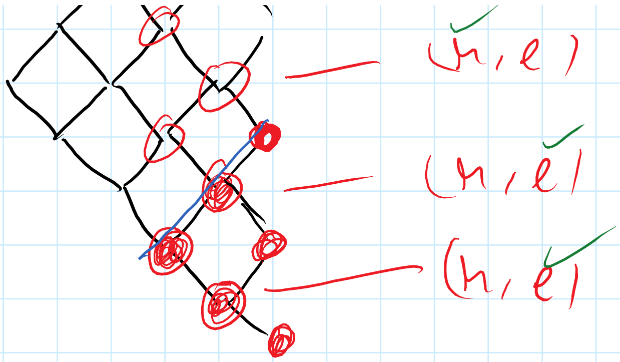
$$p_0 = \sup_{\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{Q}}\left[(K-A_\tau)_+\, e^{-r\tau}\right]$$

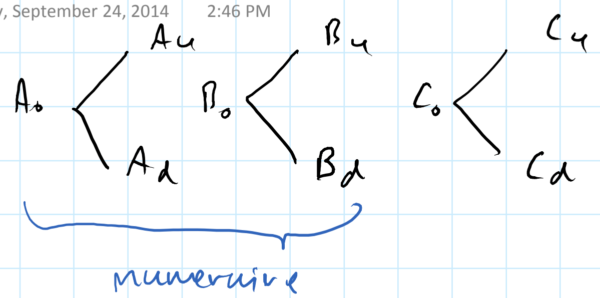
max (hold ; exercise) $= (K-A)_+$











$$\frac{C_0}{A_0} = q^a \frac{C_u}{A_u} + (1-q^a)\frac{C_d}{A_d}$$

no arb so q^a ✓

$$\left(\frac{B_0}{A_0} = q^a \frac{B_u}{A_u} + (1-q^a)\frac{B_d}{A_d} \right)$$

$$\frac{C_0}{B_0} = q^b \frac{C_u}{B_u} + (1-q^b)\frac{C_d}{B_d}$$

q^b ✓

$$\left(\frac{A_0}{B_0} = q^b \frac{A_u}{B_u} + (1-q^b)\frac{A_d}{B_d} \right)$$

$$C_0 = q^a \frac{C_u}{A_u/A_0} + (1-q^a)\frac{C_d}{A_d/A_0}$$

$$\left(\hookrightarrow C_0 = (q^b)\frac{C_u}{B_u/B_0} + (1-q^b)\frac{C_d}{B_d/B_0} \right)$$

$$= \underbrace{\left(q^a \frac{B_u/B_0}{A_u/A_0}\right)}_{q^* > 0} \frac{C_u}{B_u/B_0} + \underbrace{\left((1-q^a)\frac{B_d/B_0}{A_d/A_0}\right)}_{q^{**} > 0} \frac{C_d}{B_d/B_0}$$

$$q^* + q^{**} = \Big(\underbrace{q^a \frac{B_u}{A_u} + (1-q^a)\frac{B_d}{A_d}}_{\hookrightarrow\ B_0}\Big)\frac{A_0}{B_0} = 1$$

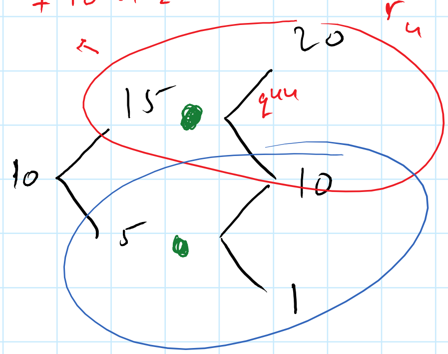
$\hookrightarrow \frac{B_0}{A_0}$

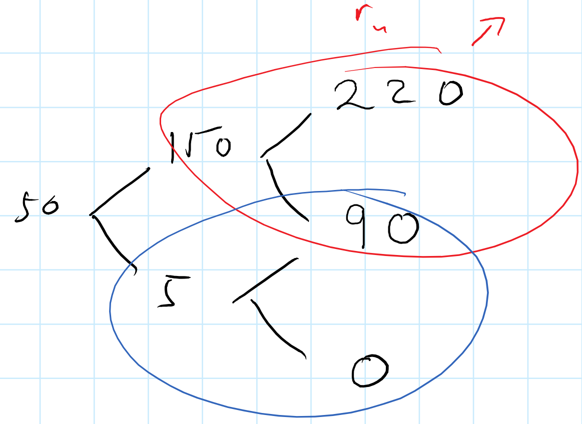
so: $$\boxed{\begin{aligned} q^b &= q^a \cdot \frac{B_u / B_0}{A_u / A_0} \\ (1-q^b) &= (1-q^a) \, \frac{B_d / B_0}{A_u / A_0} \end{aligned}}$$

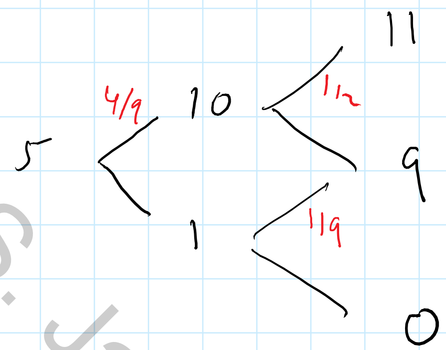
$$\boxed{\mathbb{Q}^B(\omega) = \mathbb{Q}^A(\omega) \cdot \left(\frac{B_1 / B_0}{A_1 / A_0}\right)(\omega)}$$

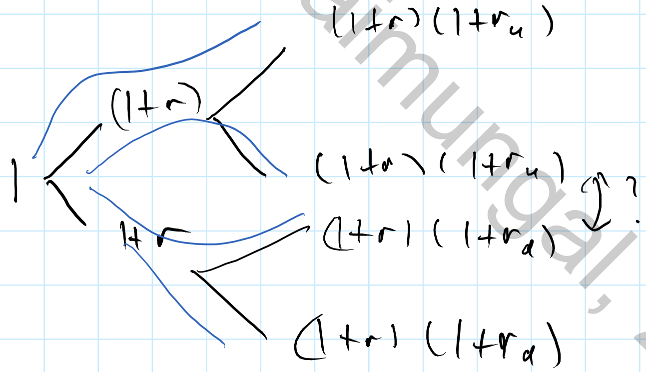
$15 = (20 q^{uu} + 10(1-q^{uu}))/(1+r_u)$

$$150 = \frac{(220 q^{uu} + 90(1-q^{uu}))}{1+r_u}$$









$r_1 \downarrow$

$$r_0 \xrightarrow{q} r_u \to r_{uu}, \quad r_{ud} = r_{du}, \quad r_d \to r_{dd}$$

r_n applies over $[t_n, t_{n+1}]$

$$P_0(1) \to 1, \ 1$$

$$\frac{P_0(1)}{1} = \frac{1}{1+r_0} q + \frac{1}{1+r_0}(1-q)$$

$$\Rightarrow \quad P_0(1) = (1+r_0)^{-1}$$

$$P_0(2) \xrightarrow{q} P_{1u}(2), \ P_{1d}(2) \to 1, \ 1, \ 1$$

$$P_{1u}(2) = (1+r_u)^{-1}$$

$$P_{1d}(2) = (1+r_d)^{-1}$$

$$\frac{P_0(2)}{1} = q\,\frac{(1+r_u)^{-1}}{(1+r_0)} + (1-q)\,\frac{(1+r_d)^{-1}}{(1+r_0)}$$

$$\Leftrightarrow q$$

$$P_0(3) \xrightarrow{q} P_{1u}(3) \xrightarrow{q^{uu}} (1+r_{uu})^{-1}, \quad (1+r_{ud})^{-1}, \quad P_{1d}(3) \xrightarrow{q^{du}} (1+r_{dd})^{-1} \to 1, \ 1, \ 1, \ 1$$

$$P_0(3) = f(q^{uu}, q^{du})$$

$\to$ no unique result!

Bonds alone do not result in unique $\mathbb{Q}$ for short rate of interests

$\Rightarrow$ turn it on its head!

find the model for r-tree with $\mathbb{Q}$ on each node $= 1/2$.

$$r_n = r_{n-1} + \underset{\text{find}}{\theta_{n-1}} \Delta t + \underset{\text{fixed}}{\sigma} \sqrt{\Delta t}\, x_n$$

$x_1, x_2, \ldots$ iid (± 1) Bernoulli
$\mathbb{Q}(x_1 = +1) = 1/2$.

$$r_0 \begin{cases} \overset{1/2}{} \; r_0 + \theta_0 \Delta t + \sigma\sqrt{\Delta t} = r_u \\ r_0 + \theta_0 \Delta t - \sigma\sqrt{\Delta t} = r_d \end{cases}$$

$$r_{t = N\Delta t} \xrightarrow[N\to\infty]{\mathbb{Q}} \mathcal{N}\left(r_0 + \int_0^t \theta_u \, du \; ; \; \sigma^2 t \right)$$

$$\frac{P_0(T)}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{\overset{1}{P_T(T)}}{B_T} \right]$$

$B_T \to (1+\Delta t r_0)(1+\Delta t r_{\Delta t}) (1+\Delta t r_{2\Delta t}) \ldots (1+\Delta t r_{(N-1)\Delta t})$

$$\log(B_T) = \sum_{n=0}^{N-1} \log(1 + \Delta t \, r_{n\Delta t})$$

$\log(1+x) \sim x$

$$= \sum_{n=0}^{N-1} \left(r_{n\Delta t} \Delta t + o(\Delta t) \right)$$

$$\xrightarrow[N\to+\infty]{} \int_0^t r_u \, du$$

$$\Rightarrow \quad P_0(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t r_u \, du} \right]$$

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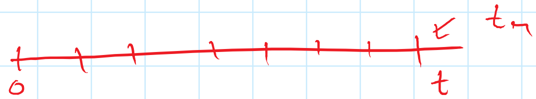
Wednesday, September 24, 2014 4:15 PM



$$r_n = r_{n-1} + \theta_{n-1}\,\Delta t + \sigma\sqrt{\Delta t}\; x_n$$

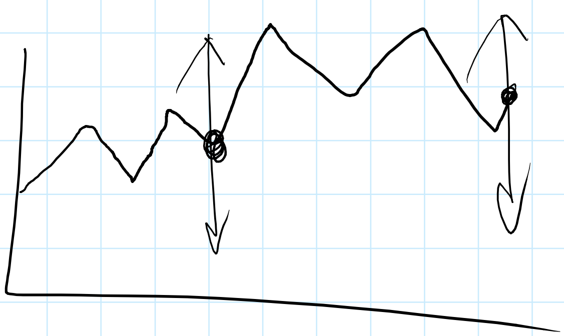
$$= \underbrace{r_0 + \sum_{m=1}^{n} \theta_{m-1}\,\Delta t}_{\xrightarrow[n\to+\infty]{} \; r_0 + \int_0^t \theta_u\, du} + \sigma\sqrt{\Delta t}\; \underbrace{\sum_{m=1}^{n} x_m}_{X}$$

$\mathbb{Q}(x_1 = +1) = 1/2$, so CLT $\Rightarrow$ $X \xrightarrow[n\to+\infty]{} \mathcal{N}(0, \sigma^2 t)$

$$\mathbb{E}^{\mathbb{Q}}[X] = 0$$

$$\mathbb{V}^{\mathbb{Q}}[X] = \sigma^2 \Delta t \sum_{m=1}^{n} \mathbb{V}^{\mathbb{Q}}[x_m]$$

$$= \sigma^2 \Delta t\, n \cdot \mathbb{V}^{\mathbb{Q}}[x_1] = \sigma^2 t$$



$$\sum_{n=1}^{N} r_{(n-1)\Delta t}\, \Delta t$$

$$= \underbrace{\sum_{n=1}^{N} \Big(r_0 + \sum_{m=1}^{n-1} \theta_{m-1}\,\Delta t}_{A} + \overbrace{\sigma\sqrt{\Delta t} \sum_{m=1}^{n-1} x_m}^{B} \Big)\, \Delta t$$

r_0 $\qquad r_0$

r_1 $\quad + \quad r_0 + \theta_0\, \Delta t + \sigma\sqrt{\Delta t}\, x_1$

$r_2 \quad \perp \quad r_0 + (\theta_0 + \theta_1)\Delta t + \sigma\sqrt{\Delta t}\,(x_1 + x_2)$

$r_3 \quad \perp \quad r_0 + (\theta_0 + \theta_1 + \theta_2)\Delta t + \sigma\sqrt{\Delta t}\,(x_1 + x_2 + x_3)$

$\vdots \qquad \vdots \qquad \vdots$

$$A \xrightarrow[N\to\infty]{} r_0\, t + \int_0^t \int_0^u \theta_s\, ds\, du$$

$$B = \left(\sum_{n=1}^{N} \sum_{m=1}^{n-1} x_m\right) \sigma (\Delta t)^{3/2}$$

$$= \sum_{n=1}^{N-1} (N-n)\, x_n\, \sigma (\Delta t)^{3/2} \xrightarrow[N\to\infty]{\mathbb{Q}} \mathcal{N}(\cdot,\cdot)$$

$$\mathbb{E}^{\mathbb{Q}}[B] = 0$$

$$\mathbb{V}^{\mathbb{Q}}[B] = \sigma^2 \Delta t^3 \sum_{n=1}^{N-1} (N-n)^2 = \sigma^2 \Delta t^3 \sum_{n=1}^{N-1} n^2$$

$$= \frac{N(2N+1)(N+1)}{6} \frac{\sigma^2\, t^3}{N^3} \to \frac{1}{3}\sigma^2 t^3$$

$$\Rightarrow \sum_{n=1}^{N} r_{(n-1)\Delta t}\, \Delta t \xrightarrow[N\to+\infty]{\mathbb{Q}} \mathcal{N}\left(r_0 t + \int_0^t \int_0^u \theta_s\, ds \,;\ \frac{1}{3}\sigma^2 t^3 \right)$$

$$\boxed{\begin{aligned} P_0(T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s\, ds} \right] \\ &= \exp\left\{ -r_0 T - \int_0^T \int_0^u \theta_s\, ds + \frac{1}{6}\sigma^2 T^3 \right\} \end{aligned}}$$

$$\boxed{\begin{aligned} P_0(T) &= \mathbb{E}^{\mathbb{Q}}\left[\, e^{-\int_0^T r_s\,ds} \right] \\ &= \exp\left\{ -r_0 T - \int_0^T \int_0^u \theta_s\,ds + \frac{1}{6}\sigma^2 T^3 \right\} \end{aligned}}$$

$$\log P_0(T) = -r_0 T - \int_0^T \int_0^u \theta_s\,ds + \frac{1}{6}\sigma^2 T^3$$

$$\partial_T \log P_0(T) = -r_0 - \int_0^T \theta_s\,ds + \frac{1}{2}\sigma^2 T^2$$

$$\partial_{TT} \log P_0(T) = -\theta_T + \sigma^2 T$$

$$\boxed{\theta_T = -\partial_{TT} \log P_0(T) + \sigma^2 T}$$

$$P_0(T) = e^{-\int_0^T f_s\,ds} = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s\,ds} \right]$$

instantaneous forward rate

$$\Rightarrow \boxed{\theta_T = \partial_T f_T + \sigma^2 T}$$

$$r_n = \overbrace{(1-\kappa\Delta t)}^{b}\, r_{n-1} + \overbrace{\kappa\,\theta_{n-1}\,\Delta t}^{a_{n-1}} + \sigma\sqrt{\Delta t}\, x_n$$

$x_1, x_2, \ldots$ iid (± 1) Bernoulli
$\mathbb{Q}(x_1) = 1/2$

$$r_n = a_{n-1} + b\, r_{n-1} + \sigma\sqrt{\Delta t}\, x_n$$

$$= a_{n-1} + b\,(a_{n-2} + b\, r_{n-2} + \sigma\sqrt{\Delta t}\, x_{n-1}) + \sigma\sqrt{\Delta t}\, x_n$$

$$= (a_{n-1} + b\, a_{n-2}) + b^2\, \overbrace{r_{n-2}}^{(a_{n-3} + b\, r_{n-3} + \sigma\sqrt{\Delta t}\, x_{n-2})} + \sigma\sqrt{\Delta t}\,(x_n + b\, x_{n-1})$$

$$= (a_{n-1} + b\, a_{n-2} + b^2 a_{n-3}) + b^3\, r_{n-3} + \sigma\sqrt{\Delta t}\,(x_n + b\, x_{n-1} + b^2 x_{n-2})$$

$$= \underbrace{\sum_{m=1}^{n} a_{n-m}\, b^{m-1}}_{\to A \xrightarrow[n\to\infty]{}} + b^n r_0 + \underbrace{\sigma\sqrt{\Delta t} \sum_{m=1}^{n+1} x_{m-1}\, b^{n-m+1}}_{\to B \xrightarrow[n\to\infty]{} \mathcal{N}(\cdot\,,\,\cdot)}$$

$$r_t \xrightarrow[n\to\infty]{} \mathcal{N}\left(r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t})\,;\ \sigma^2 \int_0^t e^{-2\kappa(t-u)}\, du \right)$$