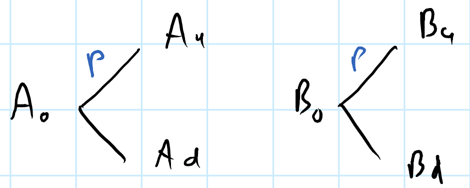


$\mathbb{P}(B_t > 0) = 1$

numeraire

$A_1 = A_u\, x + A_d\,(1-x)$, $\quad x \in \{0,1\}$ is Bernoulli

$B_1 = B_u\, x + A_d\,(1-x)$ $\qquad \mathbb{P}(x=+1) = p$

an arbitrage is trading strategy (α_t, β_t) s.t.

(α_0, β_0)

$V_0 = 0$

$\mathbb{P}(V_1 \geq 0) = 1$

$\mathbb{P}(V_1 > 0) > 0$

$$\boxed{\text{no arbitrage} \iff \exists\ \mathbb{Q} \sim \mathbb{P} \text{ s.t. } \frac{A_0}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_1}{B_1}\right]}$$

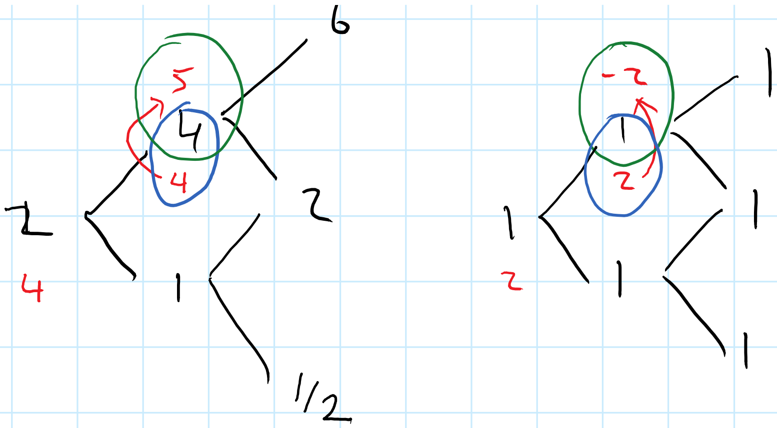
associated with the numeraire B.

$\tilde{A}_0 = \mathbb{E}^{\mathbb{Q}}[\tilde{A}_1]$

self-financing strategy $(\alpha_t, \beta_t) \in \mathcal{F}_t$

(adapted to the filtration generated by asset prices)

and $\quad \alpha_t A_{t+1} + \beta_t B_{t+1} = \alpha_{t+1} A_{t+1} + \beta_{t+1} B_{t+1}$.

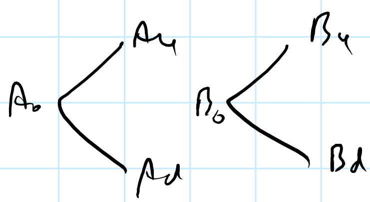


$(\alpha_0, \beta_0) = (4, 2)$, $V_0 = 10$

$(\alpha_1^u, \beta_1^u) = (5, 10)$

an arbitrage strategy is a self-financing strategy (α_t, β_t)
s.t. $V_0 = 0$
$\exists\, t$ a) $\mathbb{P}(V_t \geq 0) = 1$
b) $\mathbb{P}(V_t > 0) > 0$.

Replication of claims: trading assets to produce other assets



$C_0 \qquad C_u = \alpha A_u + \beta B_u$

$C_d = \alpha A_d + \beta B_d$

$\alpha A_0 + \beta B_0$

incomplete market: not all claims are replicable.
(number of traded assets $\geq$ number of branches (each step) $\Leftarrow$ complete)

$\Leftarrow$ $\mathbb{Q}$ is not unique

Fundamental Thm of Finance

no arbitrage $\iff$ $\exists\, \mathbb{Q} \sim \mathbb{P}$ s.t. $\forall$ traded assets A, we have

(martingale) $$\tilde{A}_s = \mathbb{E}^{\mathbb{Q}}[\tilde{A}_t \mid \mathcal{F}_s] \qquad (s<t)$$

where $\tilde{A}_t = \dfrac{A_t}{B_t}$

B_t ↳ numeraire asset.

markets are complete $\iff$ $\mathbb{Q}$ from above is unique

Wednesday, September 17, 2014 2:48 PM

CRR: $\quad A_{n+1} = A_n \, e^{\sigma\sqrt{\Delta t}\, x_n}$

$\{-1,+1\} \ni x_1, x_2, \ldots$ iid Bernoulli

$\mathbb{P}(x_1 = +1) = p = \frac{1}{2}\left(1 + \frac{\gamma}{\sigma}\sqrt{\Delta t}\right)$

$\Delta t = T/N$

$$\mathbb{E}^{\mathbb{P}}[\ln(A_T/A_0)] = \gamma . T$$

$$\mathbb{V}^{\mathbb{P}}[\ln(A_T/A_0)] \xrightarrow[N\to+\infty]{} \sigma^2 . T$$

$\Rightarrow X$ is a normal r.v.

$$\mathbb{E}^{\mathbb{P}}[e^{iuX}] \xrightarrow[N\to\infty]{} e^{iu\underbrace{\gamma T}_{\text{mean}} - \frac{1}{2}u^2 \underbrace{\sigma^2 T}_{\text{variance}}}, \qquad X = \sigma\sqrt{\Delta t}\sum_{n=1}^{N} x_n$$

$$\mathbb{E}[e^{iuZ}] = e^{-\frac{1}{2}u^2}, \qquad Z \sim \mathcal{N}(0,1)$$

$$Y = a + bZ \sim \mathcal{N}(a, b^2)$$

$$\mathbb{E}[e^{iuY}] = \mathbb{E}[e^{i(a+bZ)u}]$$

$$= e^{iau}\, \mathbb{E}[e^{i(bu)Z}]$$

$$= e^{iau}\, e^{-\frac{1}{2}b^2u^2}$$

$$\mathbb{E}[e^{iuZ}] = \int_{-\infty}^{\infty} e^{iuz}\left(\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\right) dz$$

$$= \int_{-\infty}^{\infty} e^{iuz - \frac{1}{2}z^2}\, \frac{dz}{}$$

$$= \int_{-\infty}^{\infty} e^{iuz - \frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}}$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(z - iu)^2 - (iu)^2\right]} \frac{dz}{\sqrt{2\pi}}$$

$$= e^{-\frac{1}{2}u^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-iu)^2} \frac{dz}{\sqrt{2\pi}}$$

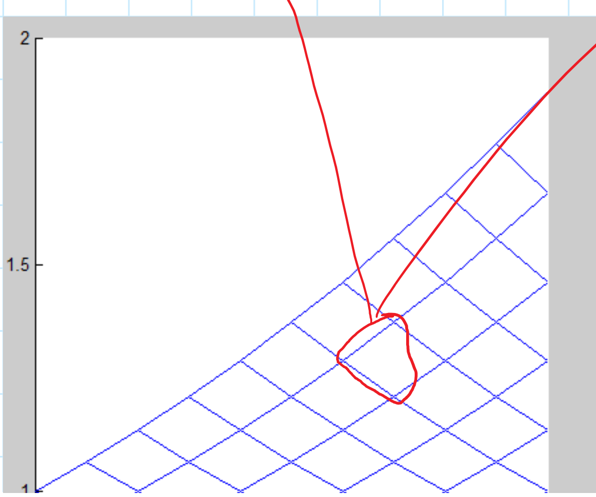
$$= e^{-\frac{1}{2}u^2} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}}}_{=1}$$

$$B_{n+1} = e^{r\Delta t} B_n, \quad A_{n+1} = e^{\sigma\sqrt{\Delta t}\, x_n} A_n$$

$$B_0 = 1$$

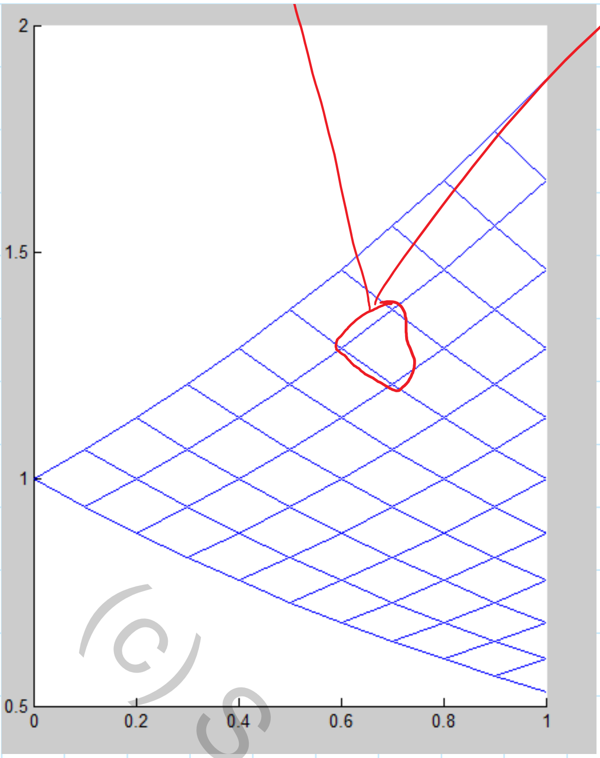
$$\mathbb{P}(x_1 = +1) = \tfrac{1}{2}\left(1 + \tfrac{\nu}{\sigma}\sqrt{\Delta t}\right)$$

$$A_{n,j} \begin{cases} \xrightarrow{q} e^{\sigma\sqrt{\Delta t}} A_{n,j} \\ \rightarrow e^{-\sigma\sqrt{\Delta t}} A_{n,j} \end{cases} \qquad e^{rn\Delta t} \begin{cases} \xrightarrow{q} e^{r(n+1)\Delta t} \\ \rightarrow e^{r(n+1)\Delta t} \end{cases}$$



$\tilde{A}$

$$e^{-rn\Delta t} A_{n,j} \begin{cases} \xrightarrow{q_{n,j}} e^{\sigma\sqrt{\Delta t} - r(n+1)\Delta t} A_{n,j} \\ \rightarrow e^{-\sigma\sqrt{\Delta t} - r(n+1)\Delta t} A_{n,j} \end{cases}$$



$\tilde{A}$

$$e^{-rn\Delta t} A_{n,j} \begin{cases} e^{\sigma\sqrt{\Delta t} - r(n+1)\Delta t} A_{n,j} \\ e^{-\sigma\sqrt{\Delta t} - r(n+1)\Delta t} A_{n,j} \end{cases}$$

$q_{n,j}$

$$e^{-rn\Delta t} \cancel{A_{n,j}} = q_{n,j}\, e^{\sigma\sqrt{\Delta t} - r(n+1)\Delta t} \cancel{A_{n,j}} + (1-q_{n,j})\, e^{-\sigma\sqrt{\Delta t} - r(n+1)\Delta t} \cancel{A_{n,j}} \qquad \left(\times e^{r(n+1)\Delta t}\right)$$

$$\Rightarrow \quad e^{r\Delta t} = q_{n,j}\, e^{\sigma\sqrt{\Delta t}} + (1-q_{n,j})\, e^{-\sigma\sqrt{\Delta t}}$$

$$\Rightarrow \quad q_{n,j} = \boxed{q = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}} \qquad \begin{matrix} E \\ F \end{matrix}$$

risk-neutral probabilities

recall $e^y = 1 + y + \frac{1}{2}y^2 + o(y^2)$

$$E = (1 + r\Delta t) - \left(1 - \sigma\sqrt{\Delta t} + \tfrac{1}{2}\sigma^2\Delta t\right) + o(\Delta t)$$

$$= \left(r - \tfrac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t} + o(\Delta t)$$

$$F = (1 + \sigma\sqrt{\Delta t} + \tfrac{1}{2}\sigma^2 \Delta t) - (1 - \sigma\sqrt{\Delta t} + \tfrac{1}{2}\sigma^2 \Delta t) + o(\Delta t)$$

$$= 2\sigma\sqrt{\Delta t} + o(\Delta t)$$

$$q \sim \frac{\sigma\sqrt{\Delta t} + (r - \tfrac{1}{2}\sigma^2)\Delta t}{2\sigma\sqrt{\Delta t}} = \frac{1}{2}\left(1 + \frac{r - \tfrac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + o(\sqrt{\Delta t})$$

recall
$$p = \frac{1}{2}\left(1 + \frac{\gamma}{\sigma}\sqrt{\Delta t}\right)$$

$$\log\frac{A_T}{A_0} \xrightarrow[N\to\infty]{\mathbb{P}} \mathcal{N}(\gamma T \,;\, \sigma^2 T)$$

$$\log\frac{A_T}{A_0} \xrightarrow[N\to\infty]{\mathbb{Q}} \mathcal{N}\left((r - \tfrac{1}{2}\sigma^2)T \,;\, \sigma^2 T\right)$$

$$\mathbb{E}^{\mathbb{P}}[A_T] = \mathbb{E}^{\mathbb{P}}[A_0 e^X]$$

$$= \mathbb{E}^{\mathbb{P}}[A_0 e^{\gamma T + \sigma\sqrt{T} Z}] \qquad Z \sim \mathcal{N}(0,1)$$

$$\left(\mathbb{E}^{\mathbb{P}}[e^{uZ}] = e^{\frac{1}{2}u^2}\right) \qquad = A_0 e^{\gamma T}\, \mathbb{E}^{\mathbb{P}}[e^{\sigma\sqrt{T} Z}]$$

$$= A_0 e^{\gamma T} e^{\frac{1}{2}\sigma^2 T}$$

$$= A_0 e^{(\gamma + \frac{1}{2}\sigma^2)T} = A_0 e^{\mu T}$$

$$\left(\gamma = \mu - \tfrac{1}{2}\sigma^2\right)$$

$$\mathbb{E}^{\mathbb{Q}}[A_T] = A_0 e^{rT}$$

$$A_0 = \frac{A_0}{} = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_T}{}\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_T}{}\right]$$

$$A_0 = \frac{A_0}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_T}{B_T}\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_T}{e^{rT}}\right]$$

$$A_T \stackrel{d}{=} A_0 \, e^{X}, \qquad X \stackrel{\mathbb{Q}}{\sim} N\left((r - \tfrac{1}{2}\sigma^2) T ;\ \sigma^2 T \right)$$

value a call option pays

$$(A_T - K)_+ \quad @\ T$$

$$\frac{C_0}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{C_T}{B_T} \,\middle|\, \mathcal{F}_0 \right]$$

$$\Rightarrow C_0 = e^{-rT} \, \mathbb{E}^{\mathbb{Q}}\left[(A_T - K)_+ \right]$$

$$= e^{-rT} \, \mathbb{E}^{\mathbb{Q}}\left[(A_0 e^{X} - K)_+ \right]$$

$$= e^{-rT} \, \underbrace{\mathbb{E}^{\mathbb{Q}}\left[(A_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} Z} - K)_+ \right]}_{E}$$

$$E = \int_{-\infty}^{\infty} (A_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} z} - K)_+ \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} \, dz$$

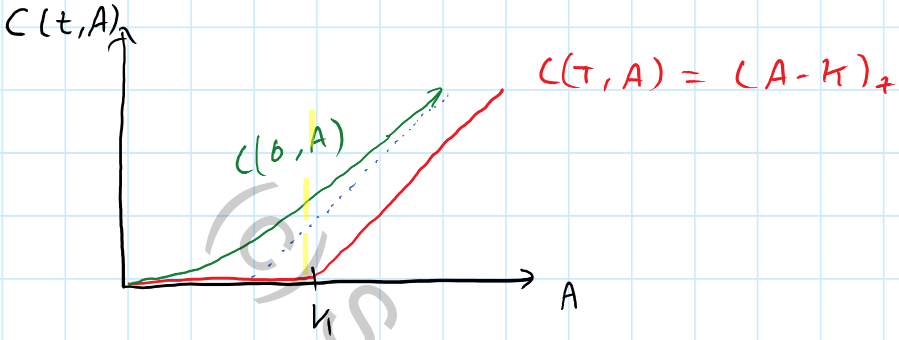
$$= \int_{z^*}^{\infty} (A_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} z} - K) \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} \, dz$$

$$A_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} z^*} - K = 0$$

$$= \cdots$$

$$\boxed{C_0 = A_0\, \Phi(d_+) - K e^{-rT}\, \Phi(d_-) \qquad d_\pm = \frac{\ln(A_0/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}$$

Black-Scholes pricing formula



$$\frac{C_0}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{(A_T - K)_+}{B_T}\right]$$

$$(A_T - K)_+ = \underbrace{A_T \mathbb{1}_{A_T > K}}_{\text{asset or nothing } N} - \underbrace{K \mathbb{1}_{A_T > K}}_{\text{digital } D}$$

$$\mathbb{1}_{\{D\}} = \begin{cases} 1 & \text{if } D \text{ holds} \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{D_0}{B_0} = \mathbb{E}^{\mathbb{Q}^B}\left[\frac{D_T}{B_T}\right] \Rightarrow D_0 = e^{-rT}\,\mathbb{E}^{\mathbb{Q}^B}\left[K \mathbb{1}_{A_T > K}\right]$$

$$\Rightarrow \boxed{D_0 = K e^{-rT} \mathbb{Q}^B(A_T > K)}$$

$$\frac{N_0}{B_0} = \mathbb{E}^{\mathbb{Q}^B}\left[\frac{N_T}{B_T}\right]$$

$$\frac{N_0}{A_0} = \mathbb{E}^{\mathbb{Q}^A}\left[\frac{N_T}{A_T}\right] \Rightarrow N_0 = A_0\,\mathbb{E}^{\mathbb{Q}^A}\left[\frac{\cancel{A_T} \mathbb{1}_{A_T > K}}{\cancel{A_T}}\right]$$

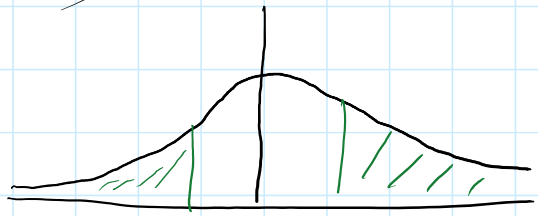
$$\Rightarrow \boxed{N_0 = A_0 \mathbb{Q}^A(A_T > K)}$$

$$\boxed{C_0 = A_0 \mathbb{Q}^A(A_T > K) - K e^{-rT} \mathbb{Q}^B(A_T > K)}$$

$$\mathbb{Q}^B(A_T > K) = \mathbb{Q}^B\left(A_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} Z} > K\right)$$

$$= \mathbb{Q}^B\left(Z > \frac{\ln(K/A_0) - (r - \tfrac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)$$

$$= \Phi\left(\frac{\ln(A_0/K) + (r - \tfrac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)$$



$$\frac{e^{rn\Delta t}}{A_n} \quad \xrightarrow{q^A} \quad \frac{e^{r(n+1)\Delta t - \sigma\sqrt{\Delta t}}}{A_n}, \qquad \frac{e^{r(n+1)\Delta t + \sigma\sqrt{\Delta t}}}{A_n}$$

$$1 = q^A \, e^{r\Delta t - \sigma\sqrt{\Delta t}} + (1 - q^A)\, e^{r\Delta t + \sigma\sqrt{\Delta t}}$$

$$q^A = \frac{e^{-r\Delta t} - e^{\sigma\sqrt{\Delta t}}}{e^{-\sigma\sqrt{\Delta t}} - e^{\sigma\sqrt{\Delta t}}}$$

$$\sim \frac{(1 - r\Delta t) - (1 + \sigma\sqrt{\Delta t} + \tfrac{1}{2}\sigma^2\Delta t)}{(1 - \sigma\sqrt{\Delta t} + \tfrac{1}{2}\sigma^2\Delta t) - (1 + \sigma\sqrt{\Delta t} + \tfrac{1}{2}\sigma^2\Delta t)} + \cdots$$

$$= \frac{-(r + \tfrac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}}{-2\sigma\sqrt{\Delta t}} + \cdots$$

$$= \frac{1}{2}\left(1 + \frac{r + \tfrac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t} \right) + \cdots$$

$$p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t} \right)$$

$$q^B = \frac{1}{2}\left(1 + \frac{r - \tfrac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t} \right) + \cdots$$

$$(r + \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z^A$$

$$A_T \overset{d}{=} A_0\, e^{(r + \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z^A}$$

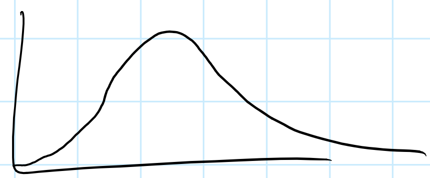
$$Z^A \overset{\mathbb{Q}^A}{\sim} \mathcal{N}(0,1)$$

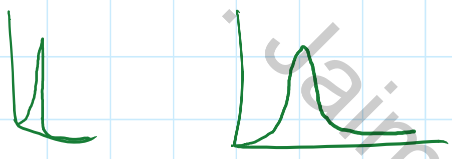
$$\Rightarrow \quad \mathbb{Q}^A(A_T > K) = \Phi\left(\frac{\log(A_0/K) + \left((r+\sigma^2) - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \right)$$

$$= \Phi\left(\frac{\log(A_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)$$

$$A_t \overset{d}{=} A_0\, e^{(r-\frac{1}{2}\sigma^2)t + \sigma\sqrt{t}\, Z}$$

$$Z \overset{\sim}{\sim} N(0,1)$$





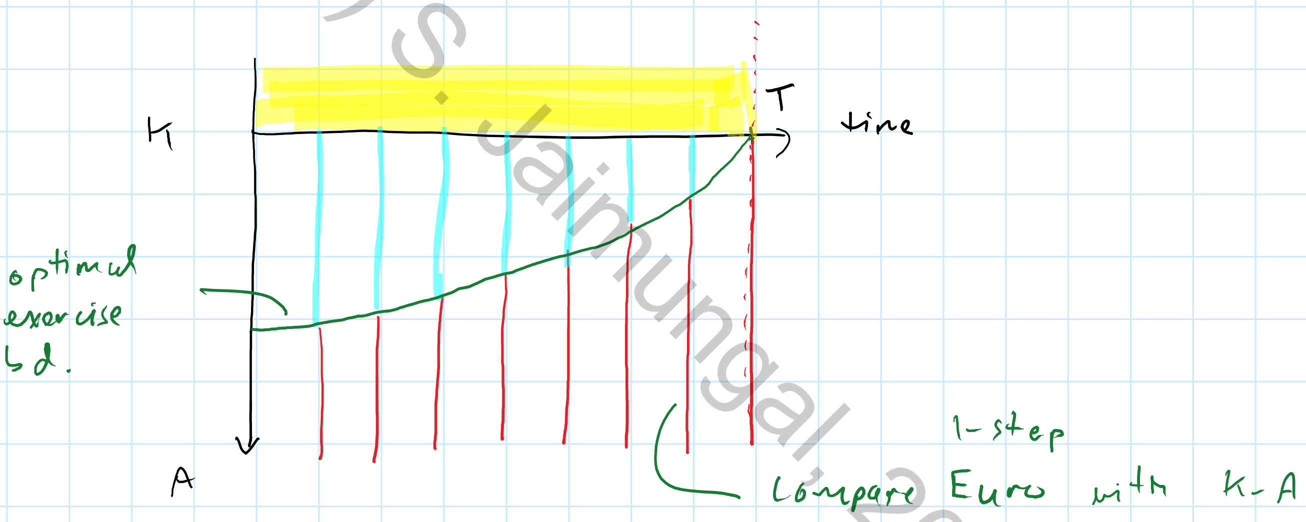
American Options

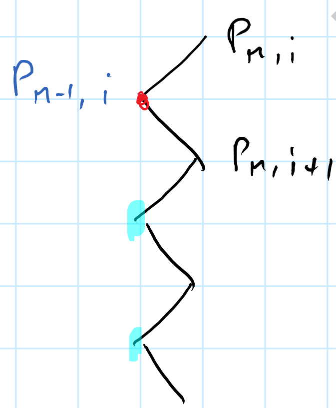
American put option : pays $(K - A_\tau)_+$ @ τ

$\tau \in (0, T]$

$$\frac{P_0}{B_0} = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}\left[\frac{(K - A_\tau)_+}{B_\tau} \right]$$

$\mathcal{T}$ set of $\mathcal{F}$ - stopping times.





hold value

$$P^h_{n-1,i} = \mathbb{E}^{\mathbb{Q}}[P_n \mid \mathcal{F}_{n-1,i}]$$

exercise value

$$P^E_{n-1,i} = (K - A_{n-1,i})_+$$

$$P_{n-1,i} = \max(P^h_{n-1,i} , P^E_{n-1,i})$$

$$A_{n+1} = e^{\sigma\sqrt{\Delta t}\, x_n} A_n$$

$q_d = (1 - e^{-\lambda\Delta t})$

$\sim \lambda \Delta t$