Feynman-Kac

30 October 2013 14:33

$$\begin{cases} \partial_t + \frac{1}{2}\partial_{xx} f = 0 \\ f(T,x) = \varphi(x) \end{cases}$$

$$\Longleftrightarrow \quad f(t,x) = \mathbb{E}[\varphi(X_T) \mid X_t = x]$$

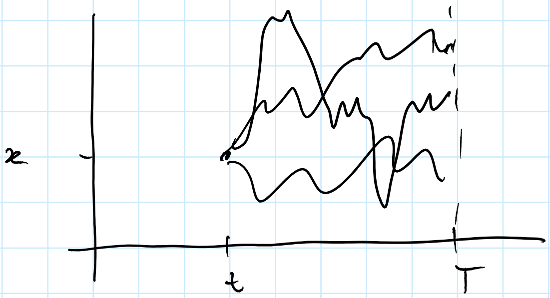
$X = (X_t)_{t \geq 0}$ is B. mtn.

$$\mathbb{E}_{t,x}[\,\cdot\,] \overset{\Delta}{=} \mathbb{E}[\,\cdot \mid X_t = x]$$

suppose

$$\begin{cases} \partial_t f + \frac{1}{2}\partial_{xx} f = 0 \\ f(T,x) = x \end{cases}$$

$f(t,x) = \mathbb{E}_{t,x}[X_T]$, X is a B.mtn



$(X_T - X_t) \sim \mathcal{N}(0, T-t)$

$\hookrightarrow x$

$\Rightarrow \quad X_T \overset{d}{=} x + \sqrt{T-t}\, Z$

$Z \sim \mathcal{N}(0,1)$

$f(t,x) = x$

$$\begin{cases} (\partial_t + \frac{1}{2}\partial_{xx}) f = 0 \\ f(T,x) = x^2 \end{cases}$$

$$f(t,x) = \mathbb{E}_{t,x}[X_T^2]$$

$$= \mathbb{E}_{t,x}[[(X_T - X_t) + \overset{x}{X_t}]^2]$$

$\hookrightarrow \sim \mathcal{N}(0, T-t)$ $\quad x^2$

$\hookrightarrow \sim \mathcal{N}(0, T-t)$

$$= \mathbb{E}_{t,x}\left[(X_T - X_t)^2 + 2\,\overset{x}{X_t}\,(X_T - X_t) + \overset{x^2}{X_t^2} \right]$$

$$= (T-t) + 0 + x^2$$

$\partial_t f = -1 \quad , \quad \partial_{xx} f = 2$

$\Rightarrow \partial_t f + \frac{1}{2}\partial_{xx} f = 0$ PDE satisfied!

$f(T,x) = x^2$ terminal condition is satisfied!

$$\begin{cases} (\partial_t + \frac{1}{2}\partial_{xx}) f = 0 \\ f(T,x) = e^{-x} \end{cases}$$

$$f(t,x) = \mathbb{E}_{t,x}\left[e^{-X_T} \right] = e^{-x + \frac{1}{2}(T-t)}$$

note: $X_T \big|_{X_t = x} \sim \mathcal{N}(x, T-t)$

$\partial_t f = -\frac{1}{2} f \quad , \quad \partial_x f = -f \quad , \quad \partial_{xx} f = f$

$\Rightarrow \partial_t f + \frac{1}{2}\partial_{xx} f = 0$ PDE satisfied.

$f(T,x) = e^{-x}$ terminal cond satisfied.

$$\begin{cases} \partial_t f + a\,\partial_x f + \frac{1}{2} b^2\,\partial_{xx} f = c f \\ f(T,x) = \varphi(x) \end{cases}$$

$$f(t,x) = \mathbb{E}_{t,x}\left[e^{-c(T-t)}\,\varphi(X_T) \right]$$

$X = (X_t)_{t \geq 0}$ is now a drifted B.motn.

$$dX_t = a\,dt + b\,dW_t$$

$\hookrightarrow$ std. B.motn.

$$dX_t = a\,dt + b\,dW_t$$

$\hookrightarrow$ std. B.mtn.

$$X_t = at + bW_t$$

$$\begin{cases} \partial_t f + a\,\partial_x f + \frac{1}{2}b^2\,\partial_{xx} f = cf \\ f(T,x) = e^{-x} \end{cases}$$

$$f(t,x) = \mathbb{E}_{t,x}\left[e^{-X_T} e^{-c(T-t)} \right]$$

B.mtn (std)

$$dX_t = a\,dt + b\,dW_t$$

$$X_T - X_t = a(T-t) + b(W_T - W_t)$$

$$\overset{d}{=} a(T-t) + b\sqrt{T-t}\,Z$$

$Z \sim N(0,1)$

$$\Rightarrow f(t,x) = \mathbb{E}_{t,x}\left[e^{-x - a(T-t) - b\sqrt{T-t}\,Z - c(T-t)} \right]$$

$$= e^{-(a+c)(T-t) + \frac{1}{2}b^2(T-t) - x}$$

$$= \exp\left\{ -x - (a + c - \tfrac{1}{2}b^2)(T-t) \right\}$$

Black-Scholes

$$\frac{dX_t}{X_t} = \mu\, dt + \sigma\, dW_t \qquad \text{traded asset (GBM)}$$

$\hookrightarrow$ $\mathbb{P}$-B.motion.

$$\frac{dM_t}{M_t} = r\, dt \qquad \text{const. IR.}$$

goal is to value a claim $f = (f_t)_{t\geq 0}$ and $f_T = \varphi(X_T)$.

Found: $F_t = f(t, X_t)$ and that fn. f satisfies the PDE:

$$\begin{cases} \partial_t f + r x\, \partial_x f + \frac{1}{2}\sigma^2 x^2\, \partial_{xx} f = r f \\ f(T,x) = \varphi(x) \end{cases}$$

$$\Rightarrow \quad f(t,x) = \mathbb{E}^{\mathbb{Q}}_{t,x}\left[e^{-r(T-t)} \varphi(X_T) \right]$$

$$dX_t = X_t r\, dt + X_t \sigma\, d\hat{W}_t \qquad \leftarrow \mathbb{Q}\text{-B.motion.}$$

$$\Leftrightarrow \quad \frac{f(t,x)}{M_t} = \mathbb{E}^{\mathbb{Q}}_{t,x}\left[\frac{f(T, X_T)}{M_T} \right]$$

$$\boxed{\frac{f_t}{M_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{f_T}{M_T} \,\Big|\, \mathcal{F}_t \right]}$$

($x \to X_t$
$\mathbb{E}_{t,x}[\cdot] \to \mathbb{E}[\cdot \mid \mathcal{F}_t]$)

$0 \to \sqrt{\Delta t}$, $-\sqrt{\Delta t}$; $\sqrt{\Delta t} \to 2\sqrt{\Delta t}$, 0; $-\sqrt{\Delta t} \to 0$, $-2\sqrt{\Delta t}$ ...

$$B_{n\Delta t} = B_{(n-1)\Delta t} + \sqrt{\Delta t}\, x_n$$

$\uparrow$ iid Bernoullis

$$\mathbb{P}(x_n = \pm 1) = \tfrac{1}{2}$$

$$(B_t)_{t\geq 0} \xrightarrow{\Delta t \downarrow 0} \text{B. motion.}$$

$$A_n \to \begin{cases} A_n\, e^{\sigma\sqrt{\Delta t} + \mu \Delta t} \\ A_n\, e^{-\sigma\sqrt{\Delta t} + \mu \Delta t} \end{cases} \qquad \frac{A_{n+1}}{A_n} = e^{\sigma \Delta B_n}$$

$$\varphi(X_T) = \mathbb{1}_{X_T > K} \qquad \text{binary call option}$$

(digital call option)

$$f(t,x) = \mathbb{E}^{\mathbb{Q}}_{\cdot}\left[e^{-r(T-t)} \mathbb{1}_{\cdot\cdot > K} \right]$$

$$F(t,x) = \mathbb{E}^{\mathbb{Q}}_{t,x}\left[e^{-r(T-t)} \mathbb{1}_{X_T > K} \right]$$

$$dX_t = r X_t dt + \sigma X_t d\hat{W}_t$$

$$\Rightarrow \boxed{X_T = X_t \, e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(\hat{W}_T - \hat{W}_t)}}$$

$$\frac{dX_t}{X_t} = r\,dt + \sigma\, d\hat{W}_t$$

"$(d\hat{W}_t)^2 = dt$"

$$d(\ln X_t) = \frac{1}{X_t} dX_t + \frac{1}{2}\left(-\frac{1}{X_t^2}\right) \cdot (dX_t)^2$$

$$= r\,dt + \sigma\, d\hat{W}_t - \frac{1}{2X_t^2} \cdot \sigma^2 X_t^2 \, dt$$

$$= (r - \tfrac{1}{2}\sigma^2)\, dt + \sigma\, d\hat{W}_t$$

$$\ln X_T - \ln X_t = (r - \tfrac{1}{2}\sigma^2)(T-t) + \sigma(\hat{W}_T - \hat{W}_t)$$

$$F(t,x) = e^{-r(T-t)} \, \mathbb{Q}_{t,x}(X_T > K)$$

$$= e^{-r(T-t)} \, \Phi\left(\frac{\ln(X_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right)$$

Measure Change

30 October 2013 15:52

we have that all traded assets must be $\mathbb{Q}$-mtgs.

$$\mathbb{E}_t[\cdot] \triangleq \mathbb{E}[\cdot \mid \mathcal{F}_t].$$

e.g. $\dfrac{A_t}{M_t} = \mathbb{E}_t^{\mathbb{Q}}\left[\dfrac{A_T}{M_T}\right]$, $A_t > 0$ a.s.

$$\frac{f_t}{M_t} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{f_T}{M_T}\right]$$

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define a measure $\mathbb{Q}^A$ via the R-N derivative:

$$\left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right)_T = \frac{A_T / A_0}{M_T / M_0}$$

let's check: $\mathbb{E}^{\mathbb{Q}}\left[\dfrac{A_T/A_0}{M_T/M_0}\right] = \dfrac{M_0}{A_0} \cdot \mathbb{E}^{\mathbb{Q}}\left[\dfrac{A_T}{M_T}\right] = \dfrac{M_0}{A_0} \cdot \dfrac{A_0}{M_0} = 1$

and $A_t, M_t > 0$ a.s. $\therefore$ it is a ~~valid~~ verified measure chg.

¿ is f/A a $\mathbb{Q}^A$-mtg ?

$g_t \triangleq \dfrac{f_t}{A_t}$ need to check that: $\mathbb{E}_t^{\mathbb{Q}^A}[g_s] = g_t$

$$\mathbb{E}_t^{\mathbb{Q}^A}[g_s] = \mathbb{E}_t^{\mathbb{Q}^A}\left[\frac{f_s}{A_s}\right] = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\frac{f_s}{A_s} \cdot \frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right]}$$

$$= \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\frac{f_s}{A_s} \cdot \frac{A_T/A_0}{M_T/M_0}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[\frac{A_T/A_0}{M_T/M_0}\right]} = \frac{f_t}{A_t} = g_t$$

$\hookrightarrow \dfrac{M_0}{A_0} \cdot \dfrac{A_t}{M_t}$

numerator: $\mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{E}_s^{\mathbb{Q}}\left[\dfrac{f_s}{A_s} \cdot \dfrac{A_T}{M_T}\right]\right]$

$$= \mathbb{E}_t^{\mathbb{Q}}\left[\frac{f_s}{A_s}\,\mathbb{E}_s^{\mathbb{Q}}\left[\frac{A_T}{M_T}\right]\right]$$

$$= \mathbb{E}_t^{\mathbb{Q}}\left[\frac{f_s}{A_s}\cdot\frac{A_s}{M_s}\right]$$

$$= \mathbb{E}_t^{\mathbb{Q}}\left[\frac{f_s}{M_s}\right] = \frac{f_t}{M_t}$$

$\therefore \left(\frac{f_t}{A_t}\right)_{t\geq 0}$ is a $\mathbb{Q}^A$ - mtg !

i.e.

$$\frac{f_t}{A_t} = \mathbb{E}_t^{\mathbb{Q}^A}\left[\frac{f_T}{A_T}\right] \quad \forall \text{ traded } f.$$

Hedging Futures

30 October 2013 16:17

Forwards: forward contract: obligation to purchase/sell an asset at a future time at a pre-determined price. (strike)

$$\varphi(X_T) = X_T - K$$

so value of Forward contract is:

$$F_t = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-r(T-t)} (X_T - K) \right]$$

$$= X_t - K P_t(T)$$

Futures: obligation to buy/sell an asset at a future time.

$F_t \equiv 0$!

Changes in the Futures price $F_t(T)$ are paid to contract holders.

t	$F_t(T)$	Margin Acct
0	100	0
1	101	1
2	103	2
3	99	-4
		-1

* trade in Futures, futures price $F_t(T)$ satisfies

$$dF_t(T) = \mu(t, F_t(T))\, dt + \sigma(t, F_t(T))\, dW_t$$

* trade in M.M.

$$\frac{dM_t}{M_t} = r\, dt$$

to value H which pays $\varphi(F_{T_0}(T))$, $h = (h_t)_{t \geq 0}$

$h_t = h(t, F_t(T))$

① α_t of Futures

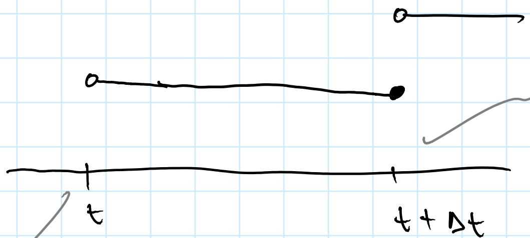
β_t of M.M.

-1 of H

$V_0 = 0$

$$V_t = \alpha_t \overset{\to 0!}{F_t} + \beta_t M_t - h_t$$

$$dV_t \underset{\text{self-financing}}{=} \alpha_t \, dF_t(T) + \beta_t \, dM_t - dh_t$$



α_t of futures (paid dividend of $\alpha_t \Delta F_t(T)$)

$\beta_t \cdot M_{t+\Delta t}$

-1 of H

α_t of futures

β_t of M.M.

-1 of H

$$\Rightarrow \quad dV_t = \alpha_t (\mu_t^F \, dt + \sigma_t^F \, dW_t) + \beta_t \, r \cdot M_t \, dt - (h_t \mu_t^h \, dt + \sigma_t^h h_t \, dW_t)$$

$$\Rightarrow \quad dV_t = \left(\alpha_t \mu_t^F + \beta_t r M_t - h_t \mu_t^h\right) dt + (\alpha_t \sigma_t^F - \sigma_t^h h_t) \, dW_t$$

remove local risk: $\boxed{\alpha_t = \dfrac{\sigma_t^M h_t}{\sigma_t^F}}$

then
we have $dV_t = (A_t)\, dt$,

$(A_t) = 0$ to avoid arbitrage

$\Rightarrow dV_t = 0 \Rightarrow V_t = 0$

$\Rightarrow \boxed{\beta_t M_t = h_t}$

so $A_t = 0 \Rightarrow \alpha_t \mu_t^F + \beta_t M_t r - h_t \mu_t^M = 0$

$\Rightarrow \alpha_t \mu_t^F - (\mu_t^M - r) h_t = 0$

$\Rightarrow \mu_t^F \sigma_t^M - (\mu_t^M - r)\sigma_t^F = 0$

$$\Rightarrow \boxed{\frac{\mu_t^F}{\sigma_t^F} = \frac{\mu_t^M - r}{\sigma_t^M}}$$

$$\begin{cases} \partial_t h + \frac{1}{2}(\sigma(t,F))^2 \partial_{FF} h = 0 \\ h(T,F) = \varphi(F) \end{cases}$$