23 October 2013 14:09

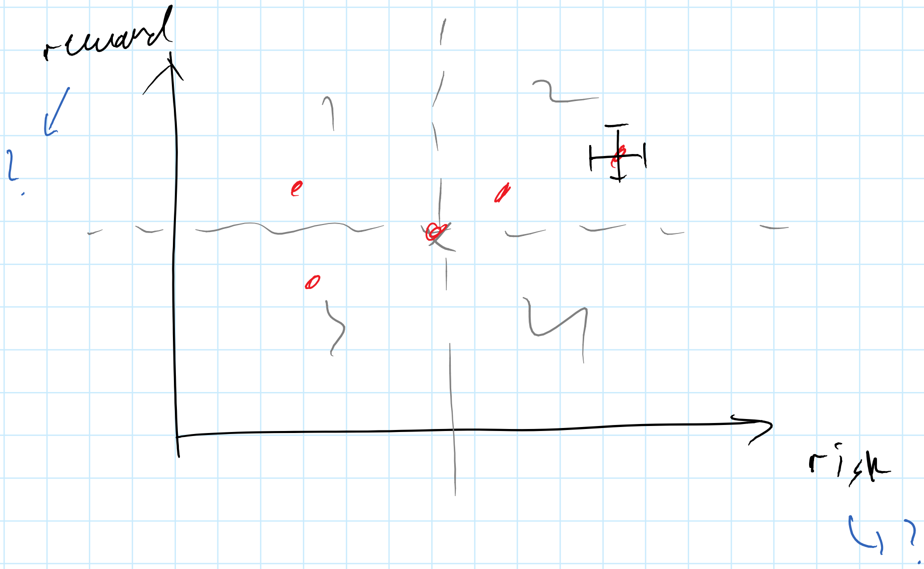
Nov 20 - tutorial

make up class Nov 28 evening

exam Dec 11 1 - 5 pm.

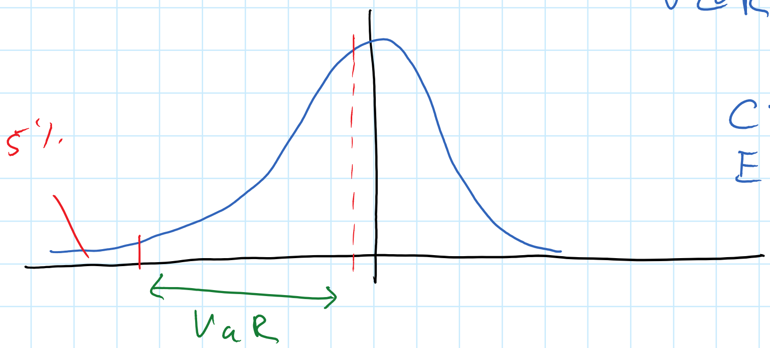
# Risk-Reward

23 October 2013 14:29




Value-at-Risk (VaR)
V@R

(~~VAR~~
~~vector auto-regressive~~)



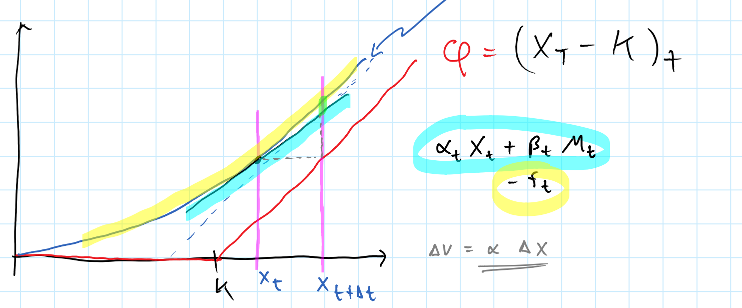

CTE - cond. tail exp.
ES - expect short fall

coherent risk measure

# Delta-Gamma Hedging

23 October 2013 15:13

$$\alpha_t = \partial_x f(t, X_t)$$




$$f(t, X_t + \Delta X) = f(t, X_t) + \Delta X \underbrace{\partial_x f(t, X_t)}_{\Delta - \text{Delta}} + \frac{1}{2}(\Delta X)^2 \underbrace{\partial_{xx} f(t, X_t)}_{\Gamma - \text{Gamma}} + \cdots$$

part of "Greeks"

Delta + Gamma locally approximate price as a quadratic fn.

## Delta-Gamma Hedge:

$\alpha_t$ - units of $X_t$

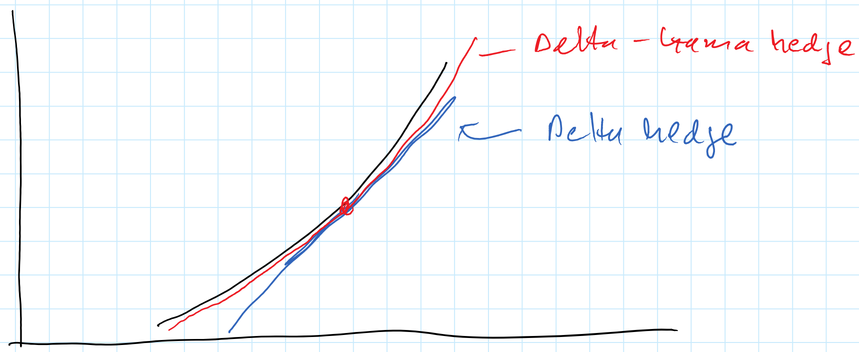
$\beta_t$ - units of $M_t$

$\gamma_t$ - units of another claim - $g_t$

$$V_t = \alpha_t X_t + \beta_t M_t + \gamma_t g_t$$

Delta: $\partial_x V_t = \alpha_t + \gamma_t \underbrace{\partial_x g}_{\Delta^g} = \Delta^f$

Gamma: $\partial_{xx} V_t = \gamma_t \underbrace{\partial_{xx} g}_{\Gamma^g} = \Gamma^f$



$$\boxed{\gamma_t = \frac{\Gamma_t^f}{\Gamma_t^g}, \qquad \alpha_t = \Delta_t^f - \frac{\Gamma_t^f}{\Gamma_t^g} \cdot \Delta_t^g}$$

@ 0 sell $f$ get $f_0$

buy $\alpha_0 = \Delta_0^f - \frac{\Gamma_0^f}{\Gamma_0^g} \cdot \Delta_0^g$ of $X$ (costs $\alpha_0 X_0$)

buy $\gamma_0 = \frac{\Gamma_0^f}{\Gamma_0^g}$ of $g$ (costs $\gamma_0 g_0$)

$M_0 = f_0 - \alpha_0 X_0 - \gamma_0 g_0$ in bank acct.

@ $t_1$: portfolio is worth: $\alpha_0 X_{t_1}$

$+ \gamma_0 g_{t_1}$

$+ M_0 e^{r\Delta t}$

rebalance to: $\alpha_{t_1}$ of $X$

$\gamma_{t_1}$ of $g$

bank acct after rebalancing is:

$$M_{t_1} = M_0 e^{r\Delta t} - (\alpha_{t_1} - \alpha_{t_0}) X_{t_1} - (\gamma_{t_1} - \gamma_{t_0}) g_{t_1}$$

in general:

$$\boxed{M_{t_k} = M_{t_{k-1}} e^{r\Delta t} - (\alpha_{t_k} - \alpha_{t_{k-1}}) X_{t_k} - (\gamma_{t_k} - \gamma_{t_{k-1}}) g_{t_k}}$$

$$\boxed{PML = M_{t_{N-1}} e^{r\Delta t} + \alpha_{t_{N-1}} X_{t_N} + \gamma_{t_{N-1}} g_{t_N} - \varphi(X_{t_N})}$$

# Self-Financing Condition

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$$V_t = \alpha_t X_t + \beta_t M_t - f_t$$

$$dV_t = \alpha_t \, dX_t + \beta_t \, dM_t - df_t$$
$$+ d\alpha_t \, X_t + d\beta_t \, M_t + d[\alpha, X]_t + d[\beta, M]_t$$

$\alpha_{t+\Delta t}, \beta_{t+\Delta t}$

$\alpha_t, \beta_t$

before rebalance | after rebalance
--- | ---

$$\alpha_t X_t + \beta_t M_t - f_t$$

before rebalance:
$$\alpha_t X_{t+\Delta t} + \beta_t M_{t+\Delta t} - f_{t+\Delta t}$$

after rebalance:
$$\alpha_{t+\Delta t} X_{t+\Delta t} + \beta_{t+\Delta t} M_{t+\Delta t} - F_{t+\Delta t}$$

$$\Delta V_t = \alpha_t \, \Delta X_t + \beta_t \, \Delta M_t - \Delta F_t$$

before = after

$$\Rightarrow \quad \Delta\alpha_t \underbrace{X_{t+\Delta t}}_{(X_t + \Delta X_t)} + \Delta\beta_t \underbrace{M_{t+\Delta t}}_{(M_t + \Delta M_t)} = 0$$

$$\Delta\alpha_t \, X_t + \Delta\alpha_t \, \Delta X_t + \Delta\beta_t \, M_t + \Delta\beta_t \, \Delta M_t = 0$$

$$\rightarrow \quad d\alpha_t \, X_t + d[\alpha, X]_t + d\beta_t \, M_t + d[\beta, M]_t = 0$$

# Feynman - Kac

Suppose $f$ satisfies the PDE:

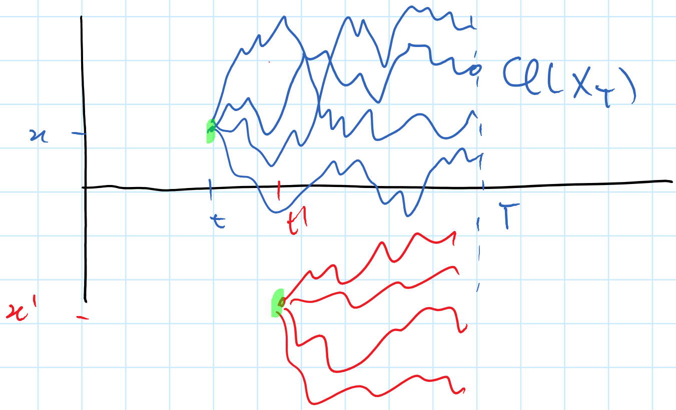
$$\begin{cases} \partial_t + \frac{1}{2} \partial_{xx} f = 0 \\ f(T, x) = \varphi(x) \end{cases}$$

Then $f$ admits the representation

$$f(t, x) = \mathbb{E}_{t,x}[\ \varphi(X_T)]$$

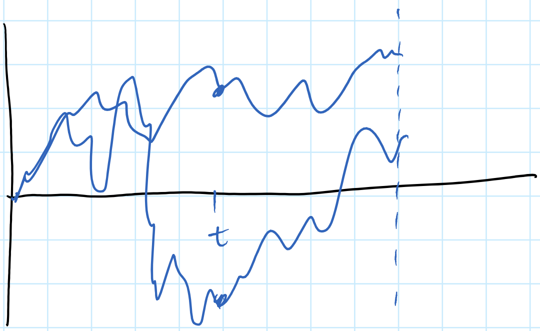
$\left( \mathbb{E}_{t,x}[\,\cdot\,] \iff \mathbb{E}[\,\cdot \mid X_t = x] \right)$

$X$ is a Brown motion.




define a stochastic process, $g_t$

$$g_t = \mathbb{E}[\ \varphi(X_T) \mid \mathcal{F}_t] = f(t, X_t)$$



$g_t$ is a Doob martingale!

$$\mathbb{E}[\, g_s \mid \mathcal{F}_t] \overset{?}{=} g_t \qquad (s>t)$$

$$\mathbb{E}[g_s \mid \mathcal{F}_t] = \mathbb{E}\Big[\underbrace{\left(\mathbb{E}[\,\varphi(X_T) \mid \mathcal{F}_s]\right)}_{\hookrightarrow g_s} \Big| \mathcal{F}_t\Big]$$

$$= \mathbb{E}[\,\varphi(X_T) \mid \mathcal{F}_t] = g_t$$

by law of iterated expectations — by defn of $g_t$!

technical condition is

$$\mathbb{E}[\,|g_t|\,] < +\infty \quad \forall\, t$$

assumed true (e.g. $\varphi(X_T) \in L^1$

$\mathbb{E}[\,|\varphi(X_T)|\,] < +\infty$ )

for $h>0$, then

Ito's lemma

$$g_{t+h} = g_t + \int_t^{t+h} (\partial_t + \tfrac{1}{2}\partial_{xx})\, g(u, X_u)\, du + \int_t^{t+h} \partial_x g(u, X_u)\, dX_u$$

$$dg_t = (\partial_t + \tfrac{1}{2}\partial_{xx})\, g(t, X_t)\, dt + \partial_x g(t, X_t)\, dX_t$$

$$\underbrace{\mathbb{E}[g_{t+h} \mid \mathcal{F}_t]}_{\hookrightarrow g_t} = g_t + \mathbb{E}\Big[\int_t^{t+h} (\partial_t + \tfrac{1}{2}\partial_{xx})\, g_u\, du \,\Big|\, \mathcal{F}_t\Big] + \underbrace{\mathbb{E}\Big[\int_t^{t+h} \partial_x g_u\, dX_u \,\Big|\, \mathcal{F}_t\Big]}_{\hookrightarrow 0\,!}$$

$$\Rightarrow \quad 0 = \mathbb{E}\left[\frac{1}{h}\int_t^{t+h} (\partial_t + \tfrac{1}{2}\partial_{xx})\, g_u \, du \;\middle|\; \mathcal{F}_t\right]$$

$$\left( \begin{aligned} & z(h) \triangleq \int_t^{t+h} l_u \, du \\ & \lim_{h \downarrow 0} \frac{z(h) - z(0)}{h} = \partial_h z \Big|_{h=0} \\ & \qquad = l_t \end{aligned} \right)$$

$$\lim_{h\downarrow 0} \Rightarrow \quad 0 = \mathbb{E}\left[ (\partial_t + \tfrac{1}{2}\partial_{xx}) g_t \mid \mathcal{F}_t \right]$$

$$\Rightarrow \quad 0 = (\partial_t + \tfrac{1}{2}\partial_{xx})\, g_t \qquad \text{true } \forall\ (t,x)$$

$$\boxed{0 = (\partial_t + \tfrac{1}{2}\partial_{xx})\, f(t,x)}$$

¿ b/c?

$$f(t,x) = \mathbb{E}[\varphi(X_T) \mid X_t = x]$$

$$f(T,x) = \mathbb{E}[\varphi(X_T) \mid X_T = x]$$

$$= \varphi(x)$$

so b/c is satisfied!

---

suppose $f$ satisfies:

$$\begin{cases} (\partial_t + \tfrac{1}{2}\partial_{xx}) f = c(t,x)\, f \\ f(T,x) = \varphi(x) \end{cases}$$

Then $f$ admits the representation:

$$f(t,x) = \mathbb{E}_{t,x}\left[\varphi(X_T)\, e^{-\int_t^T c(u, X_u)\, du}\right]$$

$X$ is a B. mtn.

$X$ is a B. mtn.

$$g_t = \mathbb{E}\left[ \varphi(X_T)\, e^{-\int_t^T c(u, X_u)\, du} \mid \mathcal{F}_t \right]$$

$$h_t = e^{-\int_0^t c(u, X_u)\, du}\, g_t$$

$$= \mathbb{E}\left[ \underbrace{\varphi(X_T)\, e^{-\int_0^T c(u, X_u)\, du}}_{\mathcal{F}_T \text{ - random variable.}} \mid \mathcal{F}_t \right]$$

$h$ is a Doob - mtg!

$$\Rightarrow \quad 0 = (\partial_t + \tfrac{1}{2} \partial_{xx}) h_t$$

$$\partial_t h_t = -c(t, X_t)\, h_t + e^{-\int_0^t c(u, X_u)\, du}\, \partial_t g_t$$

$$\partial_{xx} h_t = e^{-\int_0^t c(u, X_u)\, du}\, \partial_{xx} g_t$$

$$\Rightarrow \quad 0 = -c(t, X_t)\, g_t + (\partial_t + \tfrac{1}{2} \partial_{xx})\, g_t$$

$$\Rightarrow \quad (\partial_t + \tfrac{1}{2} \partial_{xx})\, g_t = c(t, X_t)\, g_t$$

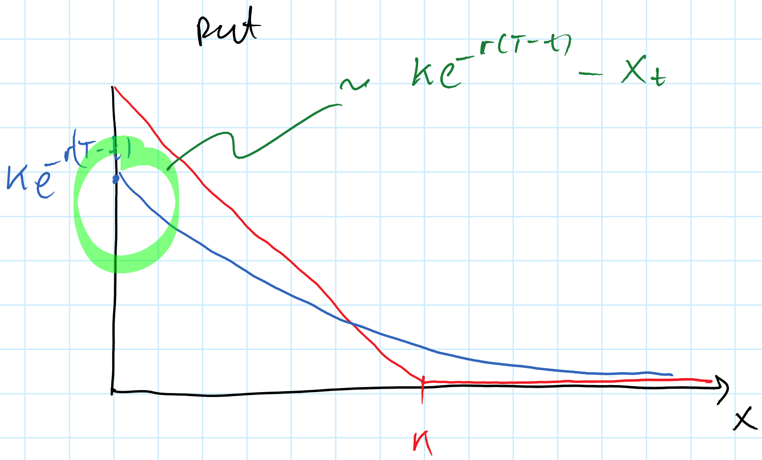
$$\Rightarrow \quad \boxed{(\partial_t + \tfrac{1}{2} \partial_{xx})\, f(t, x) = c(t, x)\, f(t, x)}$$

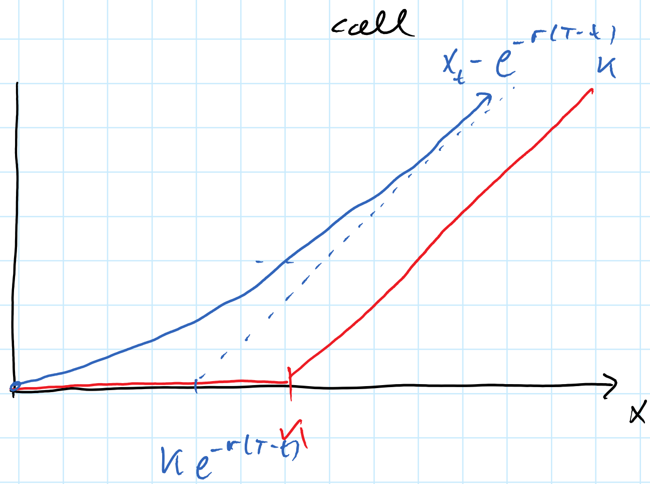
b/c: $\quad f(T, x) = \mathbb{E}[\varphi(X_T) \mid X_T = x]$

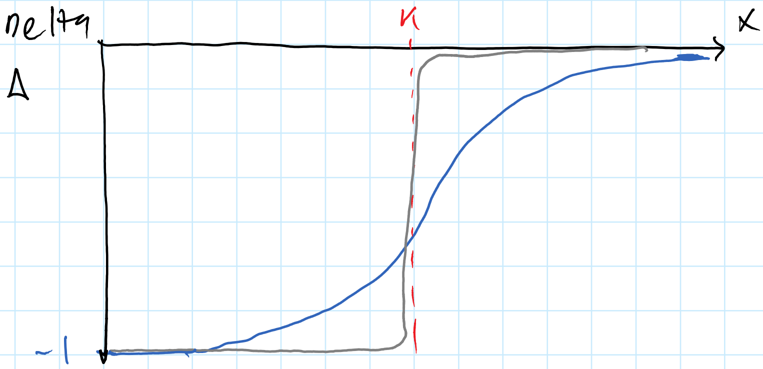
$$= \varphi(x)$$

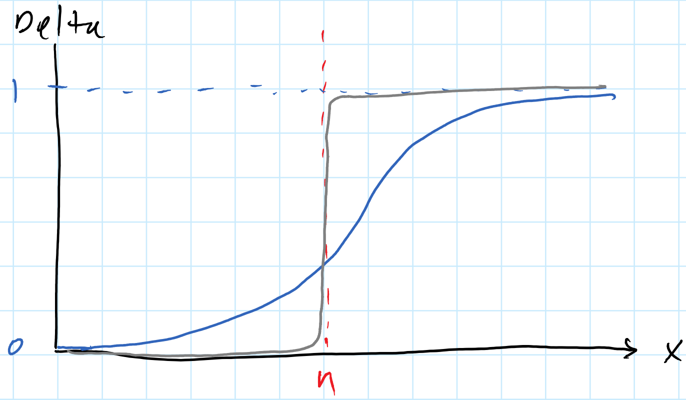
# Price, Delta & Gamma Sketches

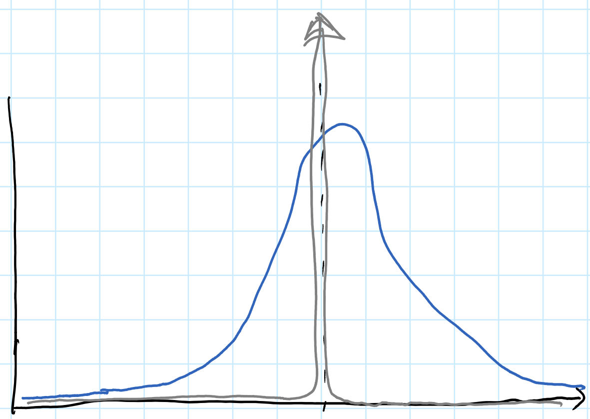
23 October 2013 16:54



$$V^{call} - V^{put} = X - Ke^{-r(T-t)}$$

$P_t(T)$

put-call parity

$$\Delta^{call} - \Delta^{put} = 1$$

$$\Gamma^{call} - \Gamma^{put} = 0 \quad !$$