Dynamic Hedging

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Black-Scholes PDE
(dynamic hedging)

* assume $\exists$ some underlying index $X = (X_t)_{0 \le t \le T}$
(not necessarily traded)

and that

$$dX_t = \underbrace{\mu^X(t, X_t)}_{\text{drift}}\, dt + \underbrace{\sigma^X(t, X_t)}_{\text{volatility}}\, dW_t$$

$\mathbb{P}$ - B. mtn

* Money market account $M = (M_t)_{0 \le t \le T}$ (traded)

$$\frac{dM_t}{M_t} = r(t, X_t)\, dt$$

* Some contingent claim on X, call this claim (traded)

$g = (g_t)_{0 \le t \le T}$ and $(g_t = g(t, X_t))$

$$\frac{dg_t}{g_t} = \mu^g(t, X_t)\, dt + \sigma^g(t, X_t)\, dW_t$$

e.g. $g(t, x) = e^{at} + bx$

$g_t = e^{at} + b X_t$

$$dg_t = \left(\partial_t g(t, X_t) + \partial_x g(t, X_t)\, \mu^X(t, X_t) + \tfrac{1}{2} \partial_{xx} g(t, X_t)\, \sigma^{X^2}(t, X_t) \right) dt + \partial_x g(t, X_t)\, \sigma^X(t, X_t)\, dW_t$$

From Ito's lemma

Goal: value a new claim $f = (f_t)_{0 \le t \le T}$

which pays $f_T = \varphi(X_T)$

$\hookrightarrow$ \phi (\varphi)

e.g. $\varphi(x) = (x-K)_+$ is a call

$f_t = f(t, X_t)$

$$\frac{df_t}{f_t} = \mu^f(t, X_t)\,dt + \sigma^f(t, X_t)\,dW_t$$

α_t - units of g_t

β_t - units of M_t

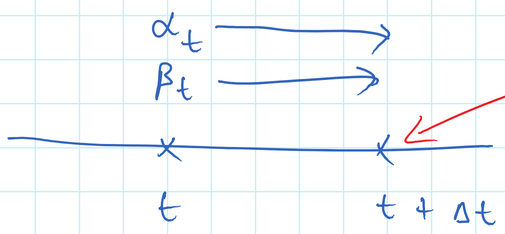
-1 - units of f_t

need <u>self-financing</u> strategies

$$V_t = \alpha_t g_t + \beta_t M_t - f_t \longleftarrow$$

need $V_0 = 0$

$$\begin{aligned} dV_t &= d(\alpha_t g_t) + d(\beta_t M_t) - df_t \\ &= d\alpha_t\, g_t + \alpha_t\, dg_t + d[\alpha, g]_t \\ &\quad + d\beta_t\, M_t + \beta_t\, dM_t + \underbrace{d[\beta, M]_t}_{\hookrightarrow 0} \\ &\quad - df_t \end{aligned}$$



$\alpha_t\, g_{t+\Delta t}$
$+ \beta_t\, M_{t+\Delta t}$
$- f_{t+\Delta t}$

$$\Delta V_t = \alpha_t (\Delta g_t) + \beta_t (\Delta M_t) - \Delta f_t$$

$$\alpha_t \, g_t + \beta_t \, M_t - f_t$$

$$dV_t = \alpha_t \, dg_t + \beta_t \, dM_t - df_t$$

└ self-financing constraint

$$= \alpha_t (\mu^g_t \, g_t \, dt + \sigma^g_t \, g_t \, dW_t) + \beta_t \, M_t \, r_t \, dt - (\mu^f_t \, f_t \, dt + \sigma^f_t \, f_t \, dW_t)$$

$$dV_t = (\alpha_t \, \mu^g_t \, g_t + \beta_t M_t \, r_t - \mu^f_t \, f_t) dt + (\alpha_t \, \sigma^g_t \, g_t - \sigma^f_t \, f_t) \, dW_t$$

locally remove risk so set

$$\boxed{\alpha_t = \frac{\sigma^f_t}{\sigma^g_t} \, \frac{f_t}{g_t}}$$

$$\Rightarrow \quad dV_t = (\alpha_t \, \mu^g_t \, g_t + \beta_t M_t \, r_t - \mu^f_t \, f_t) \; dt$$

so $dV_t = (A_t) \; dt$

and if $A_t > 0$ profit guaranteed!

if $A_t < 0$ profit " !

$$\boxed{A_t = 0 \text{ to avoid arbitrage!}}$$

so $dV_t = 0 \quad \Rightarrow \quad V_t = 0 \quad \Rightarrow \quad \alpha_t \, g_t + \beta_t M_t - f_t = 0$

$$\Rightarrow \quad \beta_t \, M_t = (f_t - \alpha_t \, g_t)$$

$$\alpha_t \, \mu^g_t \, g_t + r_t (f_t - \alpha_t \, g_t) - f_t \, \mu^f_t = 0$$

$$\alpha_t \, \mu_t^g \, g_t + r_t (f_t - \alpha_t \, g_t) - f_t \, \mu_t^f = 0$$

recall $\alpha_t = \dfrac{\sigma_t^f \, f_t}{\sigma_t^g \, g_t}$

$$\Rightarrow \quad \boxed{\frac{\mu_t^g - r_t}{\sigma_t^g} = \frac{\mu_t^f - r_t}{\sigma_t^f}} = \lambda_t = \lambda(t, X_t)$$

↳ market-price of risk

Sharpe ratio's of all assets on the underlying index are equal!

Pricing PDE

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we have that λ_t is a market property

and:

$$\frac{\mu_t^f - r_t}{\sigma_t^f} = \lambda_t$$

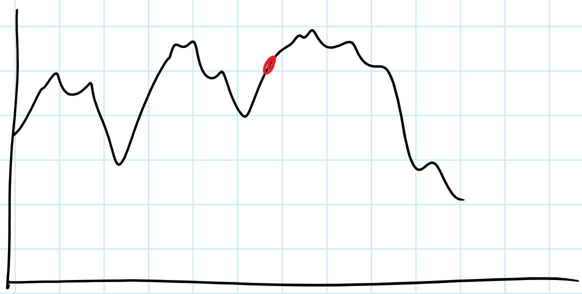
$$\Rightarrow \boxed{\mu_t^f - r_t = \lambda_t \, \sigma_t^f}$$

and recall that (via Ito's lemma)

$$\mu_t^f = \left(\partial_t f_t + \mu_t^x \, \partial_x f_t + \tfrac{1}{2}(\sigma_t^x)^2 \, \partial_{xx} f_t\right) / f_t$$

$$\sigma_t^f = \frac{\sigma_t^x \, \partial_x f_t}{f_t}$$

$$\Rightarrow \partial_t f_t + (\mu_t^x - \sigma_t^x \, \lambda_t)\, \partial_x f_t + \tfrac{1}{2}(\sigma_t^x)^2 \, \partial_{xx} f_t = r_t \, f_t$$



has to hold $\forall (t, x)$!

$$\boxed{\begin{cases} \partial_t f(t,x) + (\mu^x(t,x) - \sigma^x(t,x)\,\lambda(t,x))\,\partial_x f(t,x) \\ \qquad + \tfrac{1}{2}(\sigma^x(t,x))^2 \, \partial_{xx} f(t,x) = r(t,x)\, f(t,x), \\ \qquad f(T,x) = \varphi(x) \end{cases}}$$

generalized Black-Scholes PDE

in B-S model: $\mu^X(t,x) = \mu x \qquad \sigma^X(t,x) = \sigma x$

$$dX_t = X_t \mu \, dt + X_t \sigma \, dW_t$$

take: $g(t,x) = x$ so that X is indeed traded.

$(\sigma^g = \sigma, \; \mu^g = \mu)$ and $r(t,x) = r$ (const.)

so therefore, $\lambda = \dfrac{\mu - r}{\sigma}$

$$\Rightarrow \quad \partial_t F + \underbrace{(\mu - \sigma\lambda)}_{r} x \, \partial_x F + \tfrac{1}{2}\sigma^2 x^2 \partial_{xx} F = r F$$

$$\boxed{\begin{cases} \partial_t F + r x \, \partial_x F + \tfrac{1}{2}\sigma^2 x^2 \partial_{xx} F = r F \\ F(T,x) = \Phi(x) \end{cases}}$$

Black-Scholes PDE

e.g.

$\Phi(x) = x$ expect that $F(t,x) = x$

(i.e. the claim pays the stock value @ T)

PDE checks out!

$\Phi(x) = x^2$

From old results

$$X_T \overset{d}{=} X_t \, e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, Z}$$

$$Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

$$\text{price} = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-r(T-t)} X_T^2 \right]$$

$$= X_t^2 \, \underbrace{\mathbb{E}_t^{\mathbb{Q}}\left[e^{2(r - \frac{1}{2}\sigma^2)(T-t) + 2\sigma\sqrt{T-t}\, Z} \, e^{-r(T-t)} \right]}_{\ell(t)}$$

use an ansatz $F(t,x) = x^2\, \ell(t)$ $\quad (\ell(T) = 1)$

$$\partial_t F = x^2 \dot{\ell}, \quad \partial_x F = 2x\,\ell, \quad \partial_{xx} F = 2\ell$$

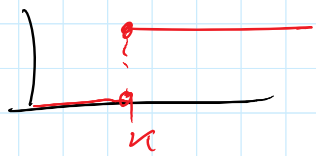
$$\Big[\underbrace{x^2 \dot{\ell}}_{\partial_t F} + \underbrace{r x (2x\ell)}_{r x\, \partial_x F} + \underbrace{\tfrac{1}{2}\sigma^2 x^2 (2\ell)}_{\frac{1}{2}\sigma^2 x^2 \partial_{xx} F}\Big] = \underbrace{r x^2 \ell}_{rF}$$

$$\Rightarrow \quad \dot{\ell} + (r + \sigma^2)\ell = 0$$

$$\ell = e^{+(r+\sigma^2)(T-t)}$$

RCLL

$$\varphi(x) = \mathbb{1}_{x \ge K} \qquad \text{digital call}$$



From old results:

$$= \text{price} = \mathbb{E}^{\mathbb{Q}}_t\big[\mathbb{1}_{X_T \ge K}\big]\, e^{-r(T-t)}$$

$$= \mathbb{Q}_t(X_T \ge K)\, e^{-r(T-t)}$$

$$= e^{-r(T-t)}\, \Phi\left(\frac{\ln(X_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$F(t,x) = e^{-r(T-t)}\, \Phi\Bigg(\underbrace{\frac{\ln(x/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}}_{d_-(t,x)}\Bigg)$$

$$\partial_t F = e^{-r(T-t)}\, \Phi'(d_-(t,x))\, \partial_t d_- + rF$$

$$\hookrightarrow \left[\frac{1}{2}\frac{\ln(x/K)}{\sigma (T-t)^{3/2}} + \frac{(r - \frac{1}{2}\sigma^2)}{2\sigma\sqrt{T-t}}\right]$$

$$\partial_x F = e^{-r(T-t)}\, \Phi'(d_-(t,x))\, \partial_x d_- \qquad \times\ r x$$

$$\hookrightarrow \frac{1}{x\sigma\sqrt{T-t}}$$

$$e^{-r(T-t)} \{ \ldots$$

$x\sigma\sqrt{T-t}$

$$\partial_{xx} F = e^{-r(\cdot)} \Big\{ \Phi''(d_-(t,x))\, \partial_x d_- + \Phi'(d_-(t,x))\, \partial_{xx} d_- \Big\}$$

$\times \frac{1}{2}\sigma^2 x^2$

$\hookrightarrow -\frac{1}{x^2\sigma\sqrt{T-t}}$

$$\Phi''(x) = x\,\Phi'(x) \Leftarrow \Phi'(x) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$$

...

Time and Move Based Delta Hedging

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$$f(t,x) = \mathbb{E}^{\mathbb{Q}}_{t,x}\left[\varphi(X_T)\, e^{-\int_t^T r_u\, du} \right]$$

$\hookrightarrow \;\; | \; X_t = x$

$$dX_t = \underbrace{(\mu_t^X - \sigma_t^X \lambda_t)}_{\to\; r_t X_t \text{ when } X_t \text{ is traded.}}\, dt + \sigma_t^X\, d\widehat{W}_t$$

$\widehat{W}_t$ is a $\mathbb{Q}$ - B.mtn.

B-S

$$dX_t = \mu X_t\, dt + \sigma X_t\, dW_t$$

$$= r X_t\, dt + \sigma X_t\, d\widehat{W}_t$$

$r_t = \text{const.} = r$

e.g. call

$$f(t,x) = x\,\Phi(d_+) - K\,\Phi(d_-)\, e^{-r(T-t)}$$

$$d_\pm = \frac{\ln(x/K) \pm (r \pm \tfrac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

discrete hedging of a continuous model

1st note: $\alpha_t = \dfrac{\sigma_t^f f_t}{\sigma_t^g g_t} = \partial_x f(t, X_t)$ $\;\;\lfloor$ if X_t is traded

@ 0 sold f : get f_0

buy α_0 of X (costs $\alpha_0 X_0$)

bank acct : $M_0 = f_0 - \alpha_0 X_0$

@ t_1 : assets now have value : $\alpha_0 X_{t_1}$

bank : $M_0 e^{r\Delta t}$

must rebalance to new α_{t_1} of X (costs $\alpha_{t_1} X_{t_1}$)

bank is now : $M_{t_1} = M_0 e^{r\Delta t} - (\alpha_{t_1} - \alpha_{t_0}) X_{t_1}$

@ t_2 : assets value : $\alpha_{t_1} X_{t_2}$

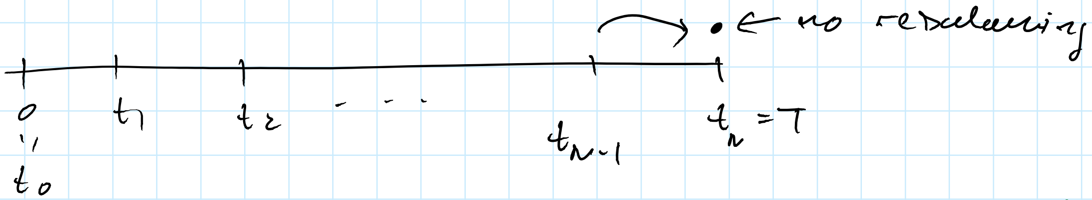
bank value : $M_{t_1} e^{r\Delta t}$

rebalance : α_{t_2} of X (costs $\alpha_{t_2} X_{t_2}$)

bank is now : $M_{t_2} = M_{t_1} e^{r\Delta t} - (\alpha_{t_2} - \alpha_{t_1}) X_{t_2}$

repeat ...

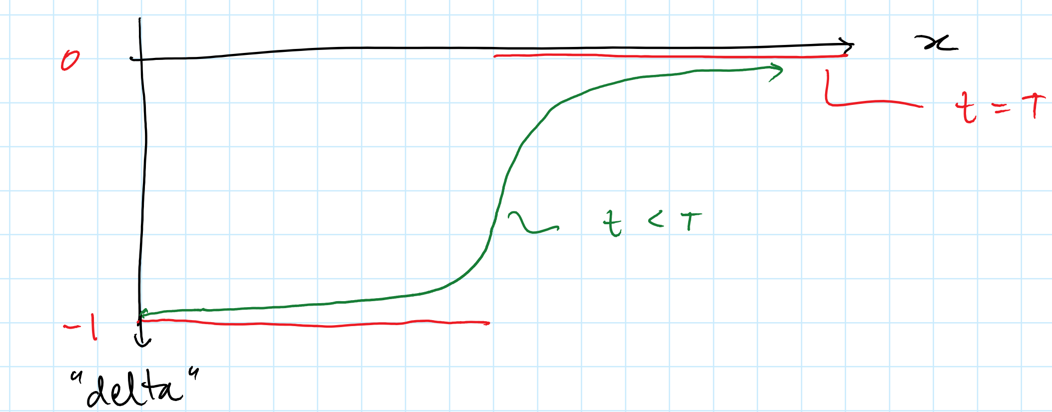
$$\boxed{M_{t_n} = M_{t_{n-1}} e^{r\Delta t} - (\alpha_{t_n} - \alpha_{t_{n-1}}) X_{t_n}}$$



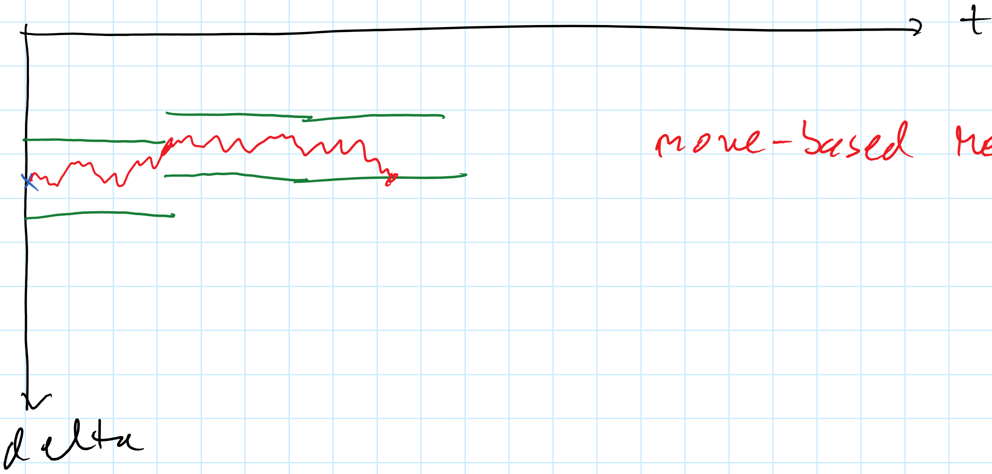
@ t_N :

owe option payoff

$$\boxed{PnL = \left(M_{t_{N-1}} e^{r\Delta t} + \alpha_{t_{N-1}} X_{t_N}\right) - \varphi(X_{t_N})}$$



α



move-based hedging