Extended Vasicek / Hull-White model

$$r_{t_n} - r_{t_{n-1}} = \kappa\left(\theta_{t_{n-1}} - r_{t_{n-1}}\right)\Delta t + \sigma\sqrt{\Delta t}\, x_n \qquad (1)$$

$\xrightarrow[N\to+\infty]{}$

$$r_T \overset{d}{=} r_0 e^{-\kappa T} + \kappa\int_0^T e^{-\kappa(T-u)}\,\theta_u\, du + \underbrace{\left(\frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa T}\right)\right)^{1/2} z}_{\sigma\sqrt{\Delta t}\sum_{m=1}^{N} x_m \beta^{N-m}}$$

(1) sum $n=1..N$

$$r_T - r_0 = \kappa\sum_{n=1}^{N}\theta_{t_{n-1}}\Delta t - \kappa\sum_{n=1}^{N} r_{t_{n-1}}\Delta t + \sigma\sqrt{\Delta t}\sum_{n=1}^{N} x_n$$

$$\Rightarrow \sum_{n=1}^{N} r_{t_{n-1}}\Delta t = \sum_{n=1}^{N}\theta_{t_{n-1}}\Delta t + \frac{r_0 - r_T}{\kappa} + \frac{\sigma\sqrt{\Delta t}}{\kappa}\sum_{n=1}^{N} x_n$$

$$(1) \Rightarrow r_T = \sum_{n=1}^{N}\alpha_{n-1}\beta^{N-n} + r_0\beta^N + \sigma\sqrt{\Delta t}\sum_{n=1}^{N} x_n\beta^{N-n}$$

$$\left(\alpha_n = \kappa\,\theta_{t_n}\Delta t\ ,\quad \beta = (1-\kappa\Delta t)\right)$$

$$\Rightarrow \sum_{n=1}^{N} r_{t_{n-1}}\Delta t = \sum_{n=1}^{N}\theta_{t_{n-1}}\Delta t + \frac{1}{\kappa}\left[r_0 - \left(\sum_{n=1}^{N}\alpha_{n-1}\beta^{N-n} + r_0\beta^N + \sigma\sqrt{\Delta t}\sum_{n=1}^{N} x_n\beta^{N-n}\right)\right] + \frac{\sigma\sqrt{\Delta t}}{\kappa}\sum_{m=1}^{N} x_m$$

$\theta_{t_{n-1}}\kappa\Delta t$

$(1-\kappa\Delta T)$

$$= \underbrace{\sum_n \theta_{t_{n-1}}\left(1-\left(1-\frac{\kappa T}{N}\right)^{N-n}\right)\Delta t}_{A}$$

$$+\ r_0\,\frac{1-\left(1-\frac{\kappa T}{N}\right)^{N}}{\kappa}$$

$$+ \; r_0 \underbrace{\frac{1-\left(1-\frac{\kappa T}{N}\right)^N}{\kappa}}_{B} \qquad A$$

$$+ \underbrace{\frac{\sigma\sqrt{\Delta t}}{\kappa} \sum_{n=1}^{N} x_n \left(1-\left(1-\frac{\kappa T}{N}\right)^{N-n}\right) \Delta t}_{X}$$

$$X \xrightarrow[N\to\infty]{d} \mathcal{N}\left(0\,;\, \frac{\sigma^2}{\kappa^2}\int_0^T \left(1-e^{-\kappa(T-u)}\right)^2 du\right)$$

$$B \xrightarrow[N\to\infty]{} \frac{1-e^{-\kappa T}}{\kappa}$$

$$A \xrightarrow[N\to\infty]{} \int_0^T \theta_u \left(1-e^{-\kappa(T-u)}\right) du$$

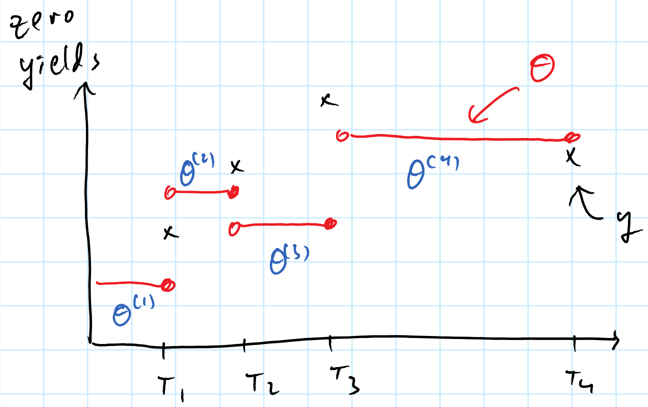
$$\Rightarrow \sum_{n=1}^{N} r_{t_{n-1}} \Delta t \xrightarrow[N\to\infty]{d} + r_0 B_T + A_T + \left(\frac{\sigma^2}{\kappa^2}\int_0^T \left(1-e^{-\kappa(T-u)}\right)^2 du\right)^{1/2} Z$$

$$Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

$$P_0(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_u\, du} \right]$$

[recall that $\mathbb{E}[e^{\alpha Z}] = e^{\frac{1}{2}\alpha^2}$, $Z \sim \mathcal{N}(0,1)$]

$$= \exp\left\{ -r_0 \frac{1-e^{-\kappa T}}{\kappa} - A_T + \underbrace{\frac{\sigma^2}{2\kappa^2}\int_0^T \left(1-e^{-\kappa(T-u)}\right)^2 du}_{C_T} \right\}$$



$$P_0(T) = e^{-T\, y(T)}$$

to match data:

$$\rightarrow \quad r_0 \frac{1-e^{-\kappa T_k}}{\kappa} + \underbrace{\int_0^{T_k} \theta_u \left(1-e^{-\kappa(T_k-u)}\right) du}_{\Theta_k} - C_{T_k} = T_k\, y^*(T_k)$$

Θ_k

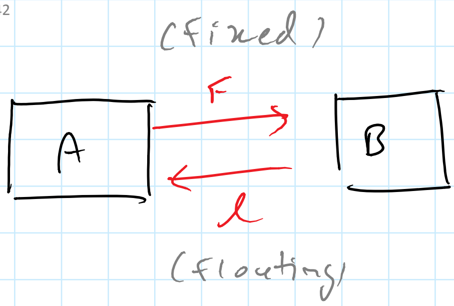
$$\Theta_k = \sum_{m=1}^{k} \int_{T_{m-1}}^{T_m} \theta^{(m)} \left(1 - e^{-\kappa (T_k - u)}\right) du$$

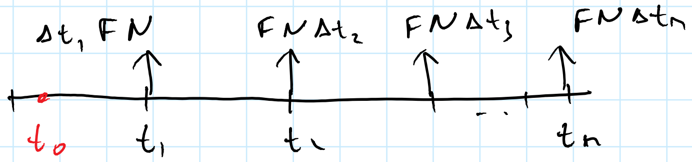
$$= \sum_{m=1}^{k} \theta^{(m)} \underbrace{\left\{ T_m - T_{m-1} - \frac{e^{-\kappa(T_k - T_m)} - e^{-\kappa(T_k - T_{m-1})}}{\kappa} \right\}}_{\alpha^{(m,k)}}$$

$$\Rightarrow \boxed{\theta^{(k)} = \frac{T_k\, y_k^* - r_0 \frac{1-e^{-\kappa T_k}}{\kappa} + C_{T_k} - \sum_{m=1}^{k-1} \theta^{(m)} \alpha^{(m,k)}}{\alpha^{(k,k)}}}$$

02 October 2013 15:42

IRS - Interest Rate Swap



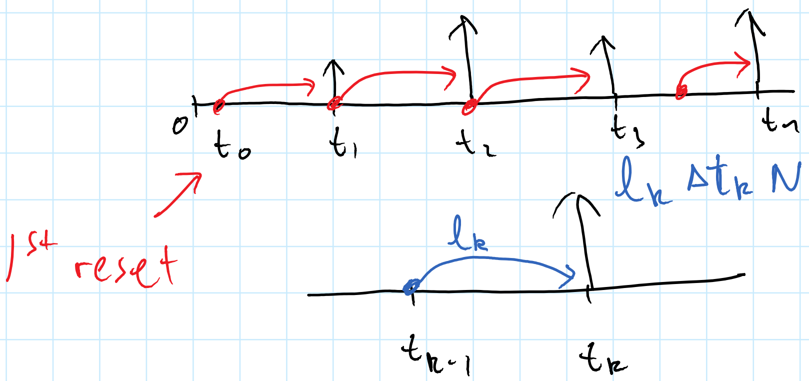


fixed-leg

$\Delta t_k = t_k - t_{k-1}$

$$V_0^{fixed} = \sum_{m=1}^{n} \Delta t_m \, F N \, P_0(t_m)$$

→ annuity



$$P_{t_{k-1}}(t_k) = (1 + \Delta t_k \cdot \ell_k)^{-1}$$

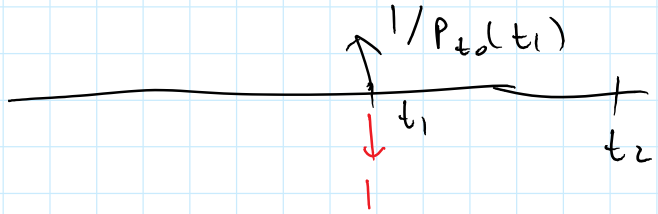
$$\Rightarrow \quad \ell_k = \frac{1}{\Delta t_k}\left(\frac{1}{P_{t_{k-1}}(t_k)} - 1\right)$$

Claim: $V_0^{Fl} = P_0(t_0) - P_0(t_n)$

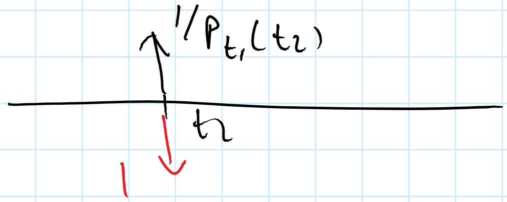
Financial argument:

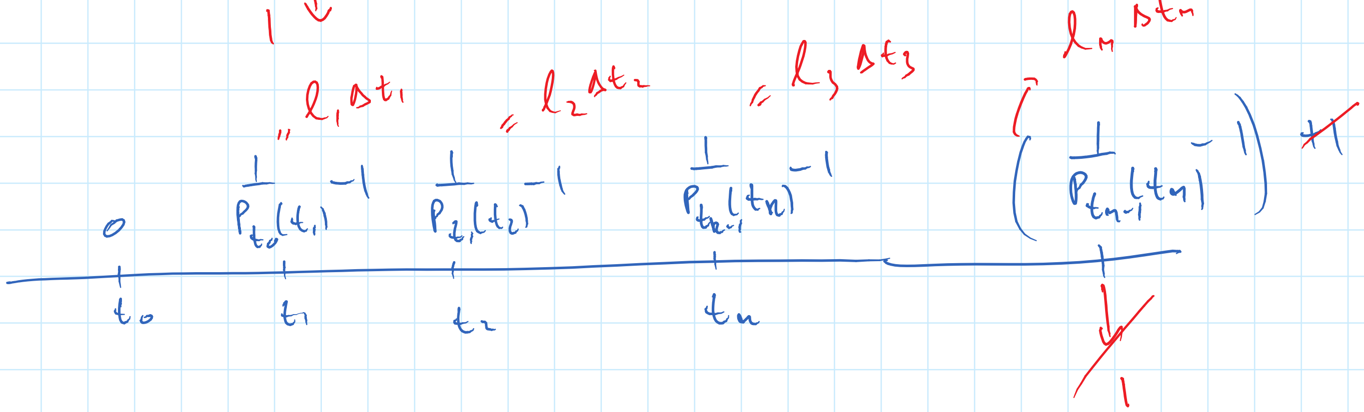
you are <u>given</u> a t_0-bond + <u>owe</u> a t_n-bond

on t_0: buy \$1 worth of t_1 bonds $\left(P_{t_0}(t_1) \right)$



on t_1: buy \$1 worth of t_2 bonds $\left(P_{t_1}(t_2) \right)$





so cash-flows generated in this self-financing strategy are equivalent to the floating-leg

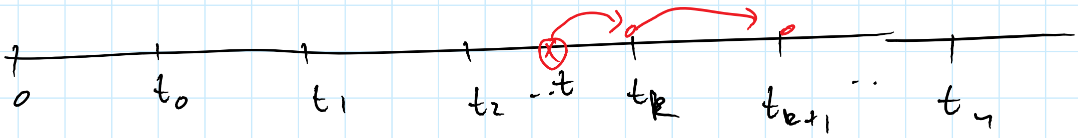
$\therefore$ have equal value.

(otherwise $\exists$ an arbitrage)

$$\Rightarrow \quad V_0^{fl} = \left(P_0(t_0) - P_0(t_n) \right) N$$

the rate F which makes $V_0^{fix} = V_0^{fl}$ is called the swap-rate.

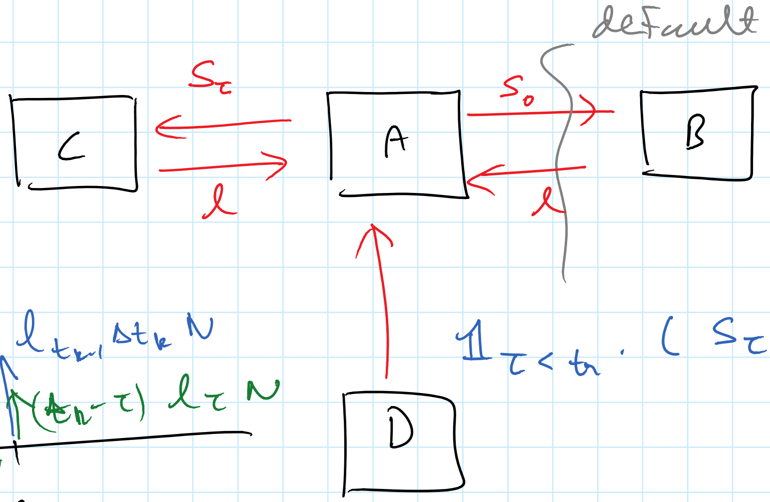
$$\Rightarrow \quad \boxed{S_0 = \frac{P_0(t_0) - P_0(t_n)}{\sum_{k=1}^{n} \Delta t_k \, P_0(t_k)}}$$



$$\boxed{S_t = \frac{1 - P_t(t_n)}{\sum_{l : t_l > t}^{n} \Delta t_l \, P_t(t_l)}}$$

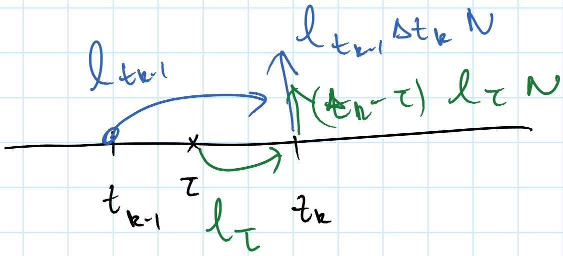
$(t_k - t)$, $(t_{k+1} - t_k)$ ...

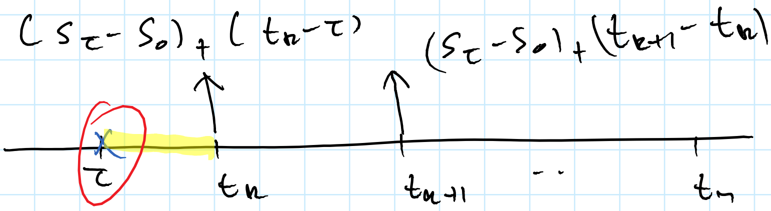
02 October 2013 16:24



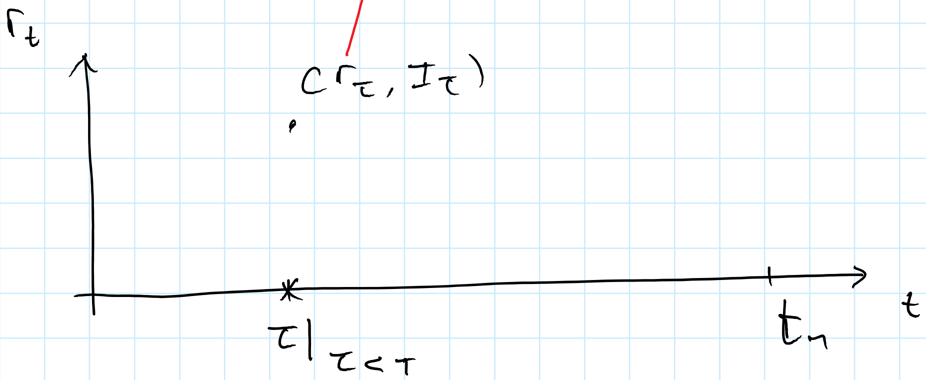
$\mathbb{1}_{\tau < t_n} \cdot (S_\tau - S_0)_+$

stream paid at payment dates





$$V^D_\tau = (S_\tau - S_0)_+ \underbrace{\sum_{\ell=1}^{n} \Delta t_\ell \cdot P_\tau(t_\ell)}_{\text{annuity}} \; N$$



$\tau \sim \exp$, intensity λ

$$P_\tau(T) = \exp\left\{ -r_\tau \frac{1 - e^{-\kappa(T-\tau)}}{\kappa} - \int_\tau^T \theta_u \left(1 - e^{-\kappa(T-u)}\right) du + \frac{\sigma^2}{2\kappa^2} \int_\tau^T \left(1 - e^{-\kappa(T-u)}\right)^2 du \right\}$$

$$V_0^D = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^\tau r_s\, ds} \; V^D_\tau \right]$$