Calibration

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FTAP: no arb $\iff \exists\ \mathbb{Q} \sim \mathbb{P}$ s.t. $\forall$ traded assets X,

$$\frac{X_t}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{X_s}{B_s} \,\Big|\, \mathcal{F}_t \right]$$

B is a traded asset, $B > 0$ a.s.
(numeraire)

$$\begin{tikzcd}[column sep=large]
 & A_{n-1} e^{\sigma\sqrt{\Delta t}} \\
A_{n-1} \arrow[ur] \arrow[r] \arrow[dr, "(\hat{\lambda},\ \lambda) \times \Delta t"'] & A_{n-1} e^{-\sigma\sqrt{\Delta t}} \\
 & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & A_{n-1} e^{c} \\
A_{n-1} \arrow[ur] \arrow[dr] & \\
 & A_{n-1} e^{-c}
\end{tikzcd}$$

$A_n = A_{n-1}\, e^{c\, x_n}$, $\quad x_1, x_2, \ldots,$ iid Bernoulli (± 1)

$\mathbb{P}(x_i = +1) = p$

Find p & c to reflect what we see in data.

$$\mathbb{E}^{\mathbb{P}}\left[\ln\left(\frac{A_T}{A_0}\right) \right] = \# \overset{\hat{\mu} T}{-40.6\%} = c \cdot N \cdot (2p-1)$$

$$\mathbb{V}^{\mathbb{P}}\left[\ln\left(\frac{A_T}{A_0}\right) \right] = \# \underset{\sigma^2 T}{54.8\%} = c^2 \cdot N \cdot \mathbb{V}[x_i]$$

$$= \mathbb{E}[x_i^2] - (\mathbb{E}[x_i])^2$$

$$= 1 - (2p-1)^2$$

$$\Rightarrow \quad c \sim \sigma\sqrt{\Delta t} \ ,$$

$$p \sim \frac{1}{2}\left(1 + \frac{\hat{\mu}}{\sigma}\sqrt{\Delta t}\right)$$

Changing between measures

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$$A_0 \begin{cases} A_u \\ A_d \end{cases} \qquad B_0 \begin{cases} B_u \\ B_d \end{cases} \qquad C_0 \begin{cases} C_u \\ C_d \end{cases}$$

$\underbrace{\qquad\qquad\qquad}_{>0 \text{ a.s.}}$

$$\frac{C_0}{A_0} = q^a \frac{C_u}{A_u} + (1-q^a) \frac{C_d}{A_d} \implies q^a = \frac{\frac{C_0}{A_0} - \frac{C_d}{A_d}}{\frac{C_u}{A_u} - \frac{C_d}{A_d}}$$

$$\frac{C_0}{B_0} = q^b \frac{C_u}{B_u} + (1-q^b) \frac{C_d}{B_d} \implies q^b = \frac{\frac{C_0}{B_0} - \frac{C_d}{B_d}}{\frac{C_u}{B_u} - \frac{C_d}{B_d}}$$

$$\frac{C_0}{B_0} = q^a \frac{A_0}{B_0} \frac{C_u}{A_u} + (1-q^a) \frac{A_0}{B_0} \frac{C_d}{A_d}$$

$$= \underbrace{\left(q^a \frac{A_0}{B_0} \frac{B_u}{A_u} \right)}_{q^* > 0} \frac{C_u}{B_u} + \underbrace{\left((1-q^a) \frac{A_0}{B_0} \frac{B_d}{A_d} \right)}_{q^{**} > 0} \frac{C_d}{B_d}$$

check: $q^* + q^{**} = \frac{A_0}{B_0} \left[q^a \frac{B_u}{A_u} + (1-q^a) \frac{B_d}{A_d} \right]$

$$= \frac{A_0}{B_0} \cdot \frac{B_0}{A_0} = 1 \quad \checkmark$$

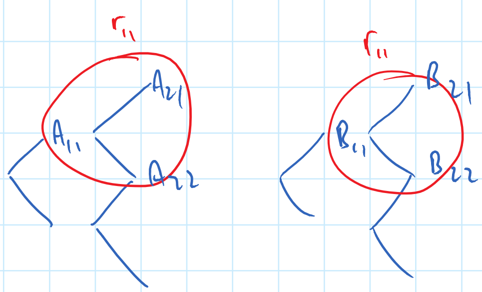
$$\therefore \quad q^b = q^a \frac{B_u / B_0}{A_u / A_0}$$

$$(1-q^b) = (1-q^a) \quad B_d / B_0$$

$$(1-q^b) = (1-q^a)\cdot\frac{B_d/B_0}{A_d/A_0}$$

$$\boxed{Q^B(\omega) = Q^A(\omega)\,\frac{B_1(\omega)/B_0}{A_1(\omega)/A_0}}$$

$$\frac{dQ^B}{dQ^A} = \frac{B_1/B_0}{A_1/A_0}$$



$$\left.\begin{aligned}
\frac{A_{11}}{1} &= \underbrace{\frac{1}{1+r_{11}}}_{e^{-r_{11}\Delta t}}\left[A_{21}\,q + A_{22}(1-q)\right] \\
\frac{B_{11}}{1} &= \frac{1}{1+r_{11}}\left[B_{21}\,q + B_{22}(1-q)\right]
\end{aligned}\right\} \Rightarrow q, r_{11}$$

Interest rate model

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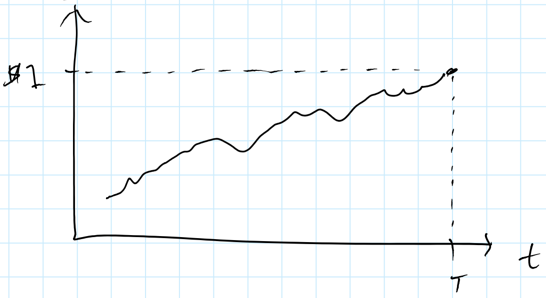
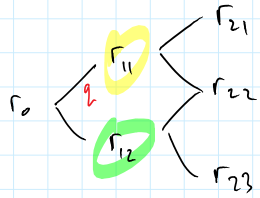
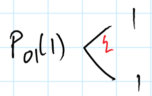
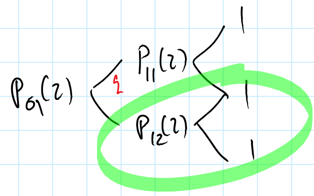
$P_t(T) \leftarrow$ zero coupon bond price







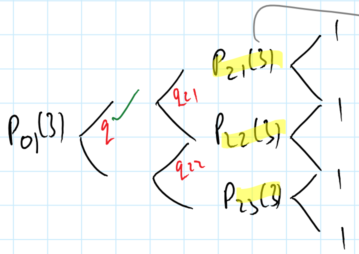
$$\frac{P_{01}(1)}{1} = \frac{1}{1+r_0} \Rightarrow r_0 = \frac{1}{P_{01}(1)} - 1$$



$$P_{11}(2) = \frac{1}{1+r_{11}}, \quad P_{12}(2) = \frac{1}{1+r_{12}}$$

$$\boxed{P_{01}(2)} = q\,\frac{P_{11}(2)}{1+r_0} + (1-q)\,\frac{P_{12}(2)}{1+r_0}$$

$\Rightarrow q$ ✓



e.g.

$$P_{2k}(3) = \frac{1}{1+r_{2k}}$$

$\Rightarrow$ not unique !

unknowns grow $\sim N^2$, eqns grow $\sim N$

Easy way around ... specify that $q = \frac{1}{2}$.

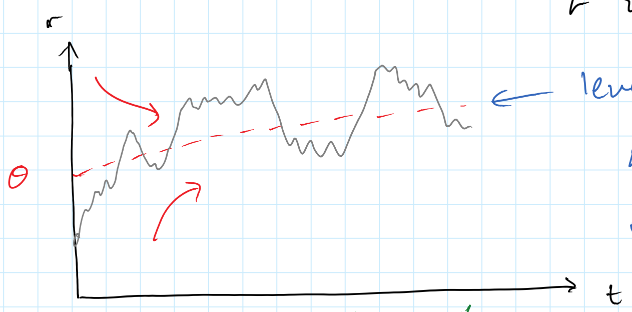
Vasicek model / AR(1)

deterministic fn of time

$$r_n - r_{n-1} = \kappa\,(\theta_{n-1} - r_{n-1})\,\Delta t + \sigma\sqrt{\Delta t}\;x_n$$

κ: kappa

$x_1, x_2, \ldots$ are iid Bernoulli ± 1, $q = \frac{1}{2}$



$\leftarrow$ level of mean-reversion

κ - rate of mean-reversion

σ - volatility.

choose as const.

$$r_n = \underbrace{\kappa\,\theta_{n-1}\,\Delta t}_{\alpha} + \underbrace{(1 - \kappa\,\Delta t)}_{\beta}\, r_{n-1} + \sigma\sqrt{\Delta t}\;x_n$$

$r_{n-1} = \alpha + \beta r_{n-2} + \sigma\sqrt{\Delta t}\;x_{n-1}$

$$= \alpha + \beta(\alpha + \beta r_{n-2} + \sigma\sqrt{\Delta t}\;x_{n-1}) + \sigma\sqrt{\Delta t}\;x_n$$

$$= \alpha + \beta(\alpha + \beta r_{n-2} + \sigma\sqrt{\Delta t}\, x_{n-1}) + \sigma\sqrt{\Delta t}\, x_n$$

$$= \alpha(1+\beta) + \beta^2 r_{n-2} + \sigma\sqrt{\Delta t}\,(x_n + \beta x_{n-1})$$

$$= \alpha(1+\beta) + \beta^2(\alpha + \beta r_{n-3} + \sigma\sqrt{\Delta t}\, x_{n-2}) + \sigma\sqrt{\Delta t}(x_n + \beta x_{n-1})$$

$$= \alpha(1+\beta+\beta^2) + \beta^3 r_{n-3} + \sigma\sqrt{\Delta t}\,(x_n + \beta x_{n-1} + \beta^2 x_{n-2})$$

$$= \cdots$$

$$= \alpha \underbrace{\sum_{m=1}^{n} \beta^{m-1}}_{\frac{1-\beta^n}{1-\beta}} + \underbrace{\beta^n}_{e^{-\kappa T}} r_0 + \underbrace{\sigma\sqrt{\Delta t} \sum_{M=1}^{n} x_m \beta^{n-m}}_{X}$$

$$\alpha \frac{1-\beta^n}{1-\beta} = \cancel{\kappa}\,\theta\,\cancel{\Delta t}\, \frac{1-(1-\kappa\Delta t)^n}{\cancel{\kappa\Delta t}} \xrightarrow[n\to+\infty]{} \theta(1-e^{-\kappa T})$$

$$(1-\kappa\Delta t)^n = \left(1-\kappa\frac{T}{n}\right)^n \xrightarrow[n\to+\infty]{} e^{-\kappa T}$$

$$\left(\int_0^T \kappa\,\theta_u\, e^{-\kappa(T-u)}\, du \right)$$

by a CLT argument $X \xrightarrow[n\to\infty]{d} \mathcal{N}(y\,;\,v)$

$$y = \mathbb{E}\Big[\sigma\sqrt{\Delta t}\sum_{m=1}^{n} \underbrace{x_m}_{3} \beta^{n-m}\Big] = 0$$

$$v = \mathbb{V}[X] = \sigma^2\,\Delta t\,.\sum_{m=1}^{n} \overset{=1}{\mathbb{V}[x_m]}\,\beta^{2(n-m)}$$

$$\mathbb{V}[x_1] = \mathbb{E}[x_1^2] - (\mathbb{E}[x_1])^2$$

$$= 1 - (2q-1)^2 = 1$$

$$\Rightarrow v = \sigma^2\,\Delta t \sum_{m=1}^{n} \beta^{2(n-m)}$$

$\beta = 1-\kappa\Delta t$

$$= \sigma^2\, \frac{1-\beta^{2n+1}}{1-\beta^2}\,\Delta t$$

$$= \sigma^2\, \frac{1-\left(1-\frac{\kappa T}{n}\right)^{2n+1}}{(\cancel{1}-(\cancel{1}-2\kappa\Delta t+\kappa^2\Delta t^2))}\,.\,\Delta t$$

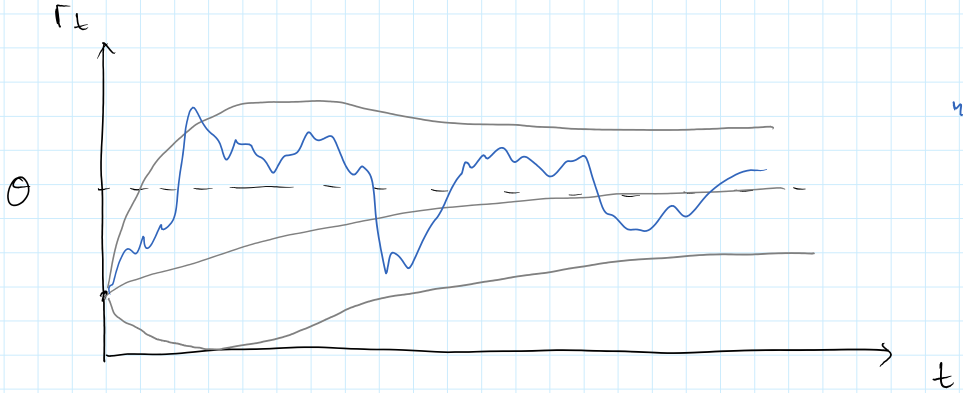
$$= \sigma^2\, \frac{1-\left(1-\frac{\kappa T}{n}\right)^{2n+1}}{} \rightarrow \frac{\sigma^2}{}\,(1-e^{-2\kappa T})$$

$$= \sigma^2 \frac{1-\left(1-\frac{\kappa T}{n}\right)^{2n+1}}{2\kappa - \kappa^2 \Delta t} \longrightarrow \frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa T}\right)$$

$$\boxed{r_T \overset{d}{=} \theta(1-e^{-\kappa T}) + r_0 e^{-\kappa T} + \sqrt{\frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa T}\right)}\; Z}$$

$$Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

note: $\mathbb{V}[r_T] \xrightarrow[T\to+\infty]{} \frac{\sigma^2}{2\kappa} < +\infty$!



note: $\mathbb{Q}(r_T < 0) > 0$.

Bond prices

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$$\frac{P_t(T)}{M_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{\overset{=1}{P_T(T)}}{M_T} \,\middle|\, \mathcal{F}_t \right]$$

$$P_t(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{M_t}{M_T} \,\middle|\, \mathcal{F}_t \right]$$

$$M_{t+\Delta t} = M_t\,(1 + r_t\,\Delta t)$$

$$M_T = \prod_{m=1}^{n} (1 + r_{m-1}\,\Delta t)\; \overset{=1}{M_0}$$

$$\ln M_T = \sum_{m=1}^{n} \ln(1 + r_{m-1}\,\Delta t)$$

$$= \sum_{m=1}^{n} r_{m-1}\,\Delta t + o(\Delta t)$$

$$\xrightarrow[n\to+\infty]{} \int_0^T r_u\,du$$

$$\boxed{M_T = e^{\int_0^T r_u\,du}}$$

$$\Rightarrow \quad P_t(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u\,du} \,\middle|\, \mathcal{F}_t \right]$$

$$\sum_{m=1}^{n} r_{m-1}\,\Delta t = ?$$

recall: $r_n - r_{n-1} = \kappa(\theta - r_{n-1})\,\Delta t + \sigma\sqrt{\Delta t}\,x_n \quad \left(\sum_{m=1}^{n} (n\to m) \right)$

$$\Rightarrow \quad r_n - r_0 = \kappa\theta T - \kappa \sum_{m=1}^{n} r_{m-1}\,\Delta t + \sigma\sqrt{\Delta t}\sum_{m=1}^{n} x_m$$

$$\hookrightarrow \quad \alpha\,\frac{1-\beta^n}{1-\beta} + r_0\,\beta^n + \sigma\sqrt{\Delta t}\sum x_m\,\beta^{n-m}$$

$$\alpha \frac{1-\beta^n}{1-\beta} + r_0 \beta^n + \sigma\sqrt{\Delta t} \sum_{m=1}^{n} x_m \beta^{n-m}$$

$$\Rightarrow \sum_{m=1}^{n} r_{m-1} \Delta t = \theta T - \frac{\alpha}{\kappa} \frac{1-\beta^n}{1-\beta} - \frac{r_0}{\kappa}\beta^n + \frac{r_0}{\kappa} + \frac{\sigma\sqrt{\Delta t}}{\kappa} \sum_{m=1}^{n} x_m (1-\beta^{n-m})$$

$$\xrightarrow[n\to+\infty]{d} \theta T - \frac{\theta}{\kappa}(1-e^{-\kappa T}) + \frac{r_0}{\kappa}(1-e^{-\kappa T}) + \zeta Z \quad , \quad Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

$$\zeta^2 = \mathbb{V}\left[\frac{\sigma\sqrt{\Delta t}}{\kappa} \sum_{m=1}^{n} x_m (1-\beta^{n-m})\right]$$

$$= \frac{\sigma^2 \Delta t}{\kappa^2} \sum_{m=1}^{n} \overset{\to 1}{\mathbb{V}[x_m]} (1-\beta^{n-m})^2$$

$$= \frac{\sigma^2}{\kappa^2} \sum_{m=1}^{n} (1-\beta^{n-m})^2 \Delta t$$

$$\hookrightarrow 1 - \frac{\kappa T}{n}$$

$$\xrightarrow[n\to+\infty]{} \frac{\sigma^2}{\kappa^2} \int_0^T (1-e^{-\kappa(T-u)})^2 \, du$$

$$\int_0^T r_u \, du \xrightarrow{d} \underbrace{\theta T + (r_0-\theta)\frac{(1-e^{-\kappa T})}{\kappa}}_{\alpha} + \zeta Z$$

$$P_0(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_u \, du}\right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[e^{-\alpha - \zeta Z}\right]$$

$$-\alpha + \frac{1}{2}\zeta^2$$

$$= e^{-\alpha + \frac{1}{2}\gamma^2}$$

$$= e^{-\text{yield } T}$$

Sample Yield

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