Review & Uniqueness

18 September 2013 14:07

FTAP: no arbitrage $\Longleftrightarrow$ $\exists\ \mathbb{Q} \sim \mathbb{P}$ s.t. $\forall$ traded assets X, we have

$$\tilde{X}_t = \mathbb{E}^{\mathbb{Q}}\left[\tilde{X}_s \mid \mathcal{F}_t\right]$$

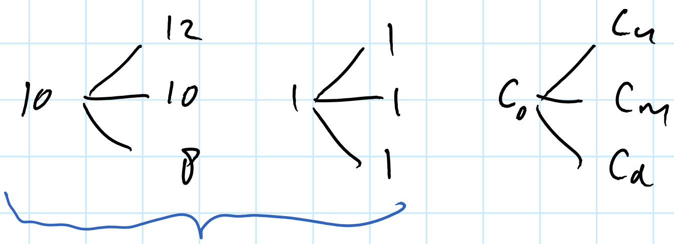
$$\tilde{X}_t = \frac{X_t}{B_t},$$ where B_t is a

numeraire asset

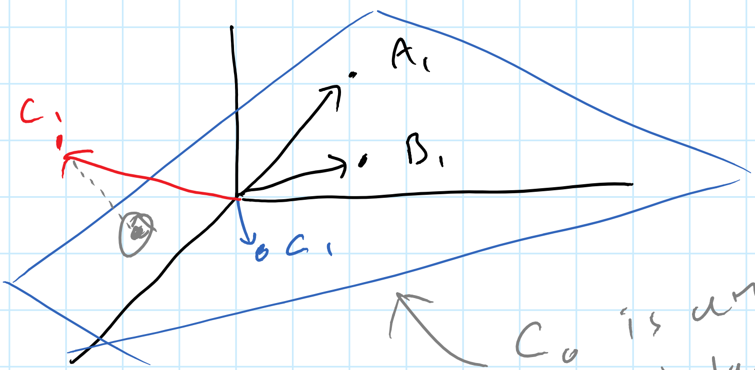
$(\ B_t > 0$ a.s.$)$

$\mathbb{Q}$ is not necessarily unique!

e.g.



$\mathbb{Q}$ is not unique $\Rightarrow$ C_0 is not necessarily unique



C_0 is unique if it lands in span(A_1, B_1)

$$\min_{(\alpha,\beta)} \mathbb{E}^{\mathbb{P}}\left[(\alpha A_1 + \beta B_1 - C_1)^2\right]$$

local risk minimizing hedge.

Default Model

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$$A_n = A_{n-1}\, e^{\sigma\sqrt{\Delta t}\, x_n}, \qquad x_1, x_2, \ldots \ \text{iid Bernoulli } (\pm 1)$$

$$\mathbb{P}(x_1 = +1) = \tfrac{1}{2}\left(1 + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) = p$$

time of default

$\tau \sim \exp$, rate $\hat\lambda$

$$\mathbb{P}(\tau \in (t, t+\Delta t] \mid \tau > t) = \left(1 - e^{-\hat\lambda \Delta t}\right) \sim \hat\lambda \Delta t$$

A_{n-1}:
- $A_{n-1} e^{\sigma\sqrt{\Delta t}}$ — $p(1-\hat\lambda\Delta t)$
- $A_{n-1} e^{-\sigma\sqrt{\Delta t}}$ — $(1-p)(1-\hat\lambda\Delta t)$
- 0 — $\hat\lambda \Delta t$

1:
- $e^{r\Delta t}$
- $e^{r\Delta t}$
- $e^{r\Delta t}$

$$\mathbb{P}(\tau > T) = e^{-\lambda T}$$

$$\mathbb{P}(\tau \in (t, t+\Delta T)) = e^{-\lambda t} - e^{-\lambda(t+\Delta t)}$$

$$= e^{-\lambda t}\left(1 - e^{-\lambda \Delta t}\right)$$

A_{n-1}:
- $A_{n-1} e^{\sigma\sqrt{\Delta t}}$ — $q\, e^{-\lambda\Delta t}$
- $A_{n-1} e^{-\sigma\sqrt{\Delta t}}$
- 0 — $(1 - e^{-\lambda\Delta t})$

λ is the $\mathbb{Q}$-intensity of default

remember when $\lambda = 0$, $$q = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

$$= \tfrac{1}{2}\left(1 + \frac{r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + o(\sqrt{\Delta t})$$

$$\frac{A_{n-1}}{B_{n-1}} = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_n}{B_n}\right] \quad \text{mtg cond.}$$

$-\lambda\Delta t$

$$\frac{A_{n-1}}{B_{n-1}} = \mathbb{E}\left[\frac{A_n}{B_n}\right] \quad \text{mtg cond.}$$

$$1 = q\overset{e^{-\lambda\Delta t}}{(1-\lambda\Delta t)} \cdot e^{\sigma\sqrt{\Delta t} - r\Delta t} + (1-q)\overset{e^{-\lambda\Delta t}}{(1-\lambda\Delta t)}\, e^{-\sigma\sqrt{\Delta t} - r\Delta t}$$

$$+ 0 \cdot \lambda\Delta t$$

$$\Rightarrow \quad q = \frac{e^{(r+\lambda)\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{+\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

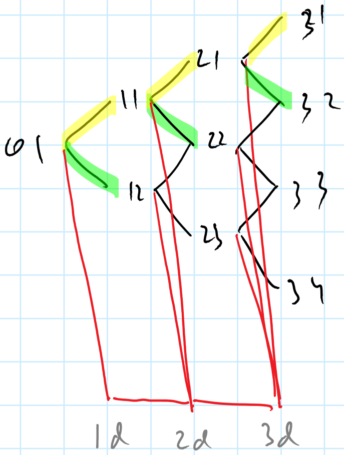
$$\sim \frac{1}{2}\left(1 + \frac{r+\lambda - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + o(\sqrt{\Delta t})$$

$$\mathbb{E}^{Q}[A_T \mid \tau > T] = A_0\, e^{(r+\lambda)T}$$

$$\mathbb{E}^{Q}[A_T] = A_0\, e^{rT}$$

European Valuation

18 September 2013 15:11



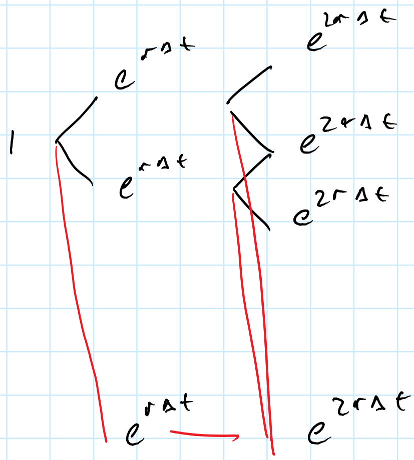
call $(A_T-K)_+$

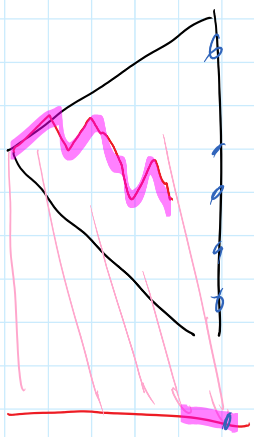
$\max(A_T-K,0)$

$$\frac{C_{n-1}}{B_{n-1}} = \mathbb{E}^{\mathbb{Q}}\left[\frac{C_n}{B_n}\right]$$

$$C_{n-1,j} = q\frac{e^{-\lambda\Delta t}}{e^{r\Delta t}} C_{n,j} + (1-q)\frac{e^{-\lambda\Delta t}}{e^{r\Delta t}} C_{n,j+1} + (1-e^{-\lambda\Delta t})\cdot C_{nd}$$

$$C_{n-1,d} = C_{n,d}\, e^{-r\Delta t}$$





$$\frac{C_0}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{C_T}{B_T}\right] \sim \frac{1}{N}\sum_{n=1}^{N} \frac{(A_T^{(n)}-K)_+}{e^{rT}}$$

$= k-(N-k) = 2k-N$

$$A_T = A_0\, e^{\sigma\sqrt{\Delta t}\sum_{n=1}^{N} x_n}\, \mathbb{1}_{\tau>T}$$

$$\binom{N}{k} \quad q^k (1-q)^{N-k}$$

$$\sigma\sqrt{\Delta t} \sum_{n=1}^{N} x_n \xrightarrow[N\to+\infty]{d} \mathcal{N}\left((r+\lambda-\tfrac{1}{2}\sigma^2)T \, ; \, \sigma^2 T \right)$$

$$A_T \xrightarrow{d} A_0 \, e^{(r+\lambda-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T} Z} \, \mathbb{1}_{\tau > T}$$

$$Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

$$\tau \underset{\mathbb{Q}}{\sim} \exp, \text{ intensity } \lambda.$$

$$\frac{C_0}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{C_T}{B_T} \right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\frac{(A_T - K)_+}{B_T} \right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\frac{A_T}{B_T} \mathbb{1}_{A_T > K} - \frac{K}{B_T} \mathbb{1}_{A_T > K} \right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\frac{A_T}{B_T} \overset{=F_T}{\mathbb{1}_{A_T > K}} \right] - \frac{K}{B_T} \mathbb{Q}(A_T > K)$$

$$\frac{F_0}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{F_T}{B_T} \right]$$

$\underline{\underline{\lambda = 0}}$: take A as the numeraire!

$$\frac{F_0}{A_0} = \mathbb{E}^{\mathbb{Q}^A}\left[\frac{F_T}{A_T} \right] = \mathbb{E}^{\mathbb{Q}^A}\left[\mathbb{1}_{A_T > K} \right]$$

$$= \mathbb{Q}^A(A_T > K)$$

$$C_0 = F_0 - K e^{-rT} \mathbb{Q}(A_T > K)$$

$$\boxed{C_0 = A_0\, \mathbb{Q}^A(A_T > K) - K e^{-rT}\, \mathbb{Q}(A_T > K)}$$

$$A_{n-1} \begin{cases} \overset{q^a}{\nearrow}\; A_{n-1} e^{\sigma\sqrt{\Delta t}} \\ \searrow\; A_{n-1} e^{-\sigma\sqrt{\Delta t}} \end{cases} \qquad\qquad 1 \begin{cases} \nearrow\; e^{r\Delta t} \\ \searrow\; e^{r\Delta t} \end{cases}$$

mtg. cond is $\quad \dfrac{1}{A_{n-1}} = q^a \dfrac{e^{r\Delta t}}{A_{n-1} e^{\sigma\sqrt{\Delta t}}} + (1-q^a) \dfrac{e^{r\Delta t}}{A_{n-1} e^{-\sigma\sqrt{\Delta t}}}$

$$\Rightarrow \quad e^{-r\Delta t} = q^a e^{-\sigma\sqrt{\Delta t}} + (1-q^a) e^{\sigma\sqrt{\Delta t}}$$

$$\Rightarrow \quad q^a = \frac{e^{-r\Delta t} - e^{\sigma\sqrt{\Delta t}}}{e^{-\sigma\sqrt{\Delta t}} - e^{\sigma\sqrt{\Delta t}}}$$

$$= \frac{(\cancel{1} - r\Delta t) - (\cancel{1} + \sigma\sqrt{\Delta t} + \tfrac{1}{2}\sigma^2 \Delta t) + \cdots}{(\cancel{1} - \sigma\sqrt{\Delta t} + \cancel{\tfrac{1}{2}\sigma^2}\Delta t) - (\cancel{1} + \sigma\sqrt{\Delta t} + \cancel{\tfrac{1}{2}\sigma^2}\Delta t) + \cdots}$$

$$= \frac{-(r + \tfrac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t} + \cdots}{-2\sigma\sqrt{\Delta t} + \cdots}$$

$$q^a = \frac{1}{2}\left[1 + \frac{r + \tfrac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right] + \cdots$$

$$\left(\text{recall } \; q = \frac{1}{2}\left(1 + \frac{r - \tfrac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + \cdots \right)$$

so $\mathbb{Q} \to \mathbb{Q} \qquad r \to r + \sigma^2$

$$\therefore \quad A_T \xrightarrow[\mathbb{Q}^a]{d} A_0\, e^{(r + \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z}$$

$$Z \underset{\mathbb{Q}^a}{\sim} N(0,1)$$

$$C_0 = A_0 \Phi(d_+) - K e^{-rT} \Phi(d_-)$$

$$d_\pm = \frac{\ln(A_0/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad \Phi(\cdot) \text{ is}$$

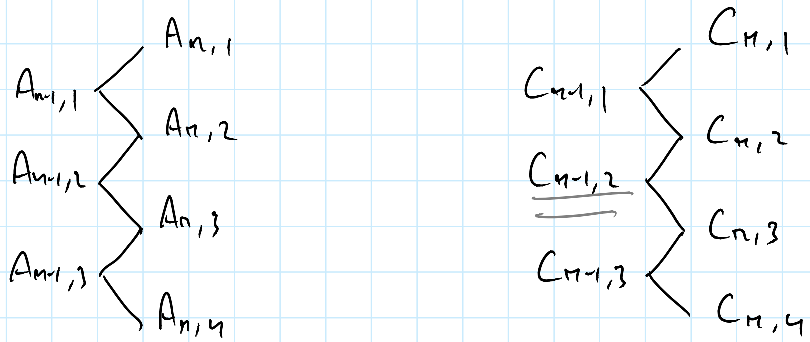
std. norm cdf.

American option lets you set the maturity up to a max date

S - stopping time bounded by T

at s $\varphi = (A_s - K)_+$

$$\frac{C_0}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{(A_s - K)_+}{B_s}\right]$$



holding value

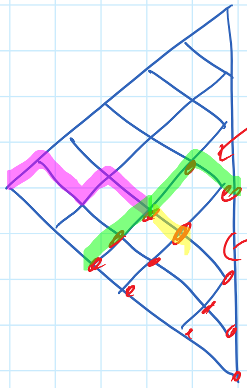
$$\frac{C^h_{n-1,k}}{B_{n-1,k}} = q\,\frac{C_{n,k}}{B_{n,k}} + (1-q)\,\frac{C_{n,k+1}}{B_{n,k+1}}$$

$$\Rightarrow C^h_{n-1,k} = e^{-r\Delta t}\left(q\,C_{n,k} + (1-q)\,C_{n,k+1}\right)$$

intrinsic value

$$C^I_{n-1,k} = (A_{n-1,k} - K)_+ \; , \; (K - A_{n-1,k})_+$$

$$C_{n-1,k} = \max\left(C^I_{n-1,k} \, , \, C^h_{n-1,k}\right)$$

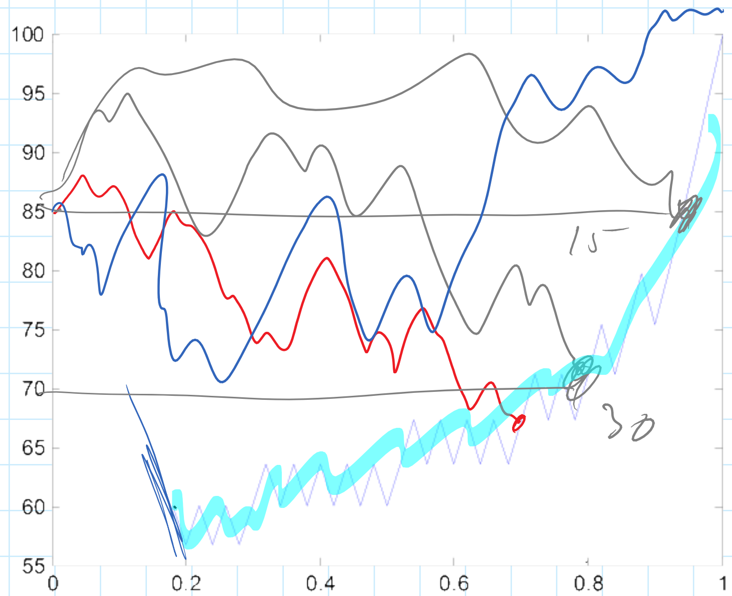


optimal exercise bd.

optimal to exercise

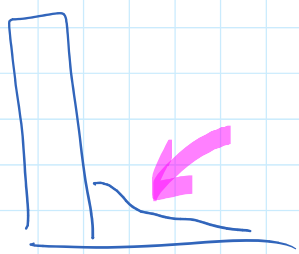
$C^I_{n-1,k} > C^H_{n-1,k}$

Sun 29, 2013
mid night



$(K - A_s)_+$

protected put gets 0 upon default.



$\mu = 10\%$

$\hat{\lambda} = 1.5\%$