27 November 2013 14:12

Office Hours:

Mon Dec 9 2 - 4 pm

Tues Dec 10 2 - 4 pm

<u>EXAM!</u>

Dec 11 12 - 4 pm

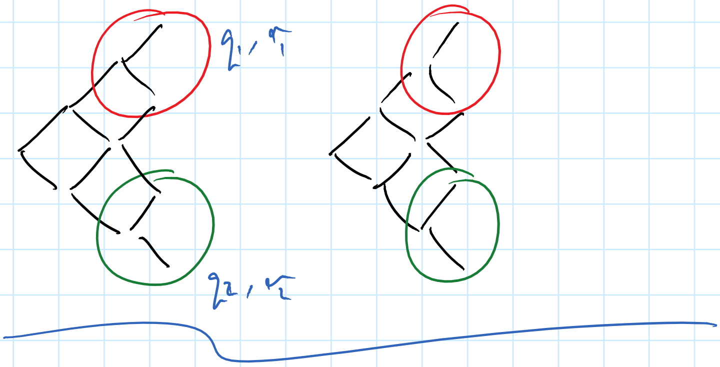
ask Ryan extra TA

Final grade : 50 - 69 } → B-

✓

$\frac{17}{26}$

Stochastic Interest Rate



$r_n = a\, r_{n-1} + b_{n-1} + x_n \sigma\sqrt{\Delta t}$ AR(1) → Vasicek model

compute bond prices → calibrate

valuing options on bonds ← difficult in closed form in discrete-time.

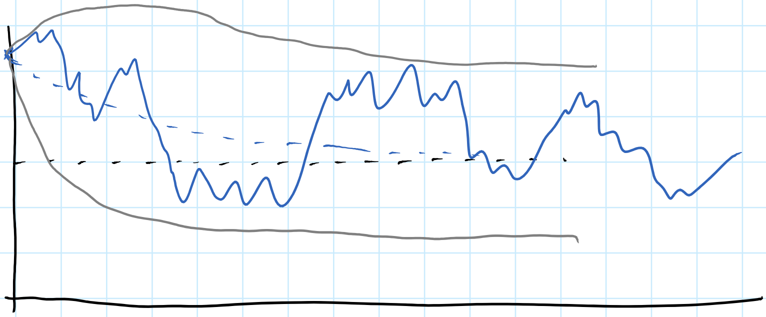
$$* \quad dr_t = \overset{\mu(t,r_t)}{\mu_t}\, dt + \overset{\sigma(t,r_t)}{\sigma_t}\, \underbrace{dW_t}_{\mathbb{P}\text{-B.motion}}$$

(short rate of interest)

e.g.: $dr_t = \kappa(\theta - r_t)\,dt + \sigma\, dW_t$ $(\kappa, \theta, \sigma > 0)$

(Vasicek model of IR, i.e. r_t is an Ornstein-Uhlenbeck)
(OU)

r is a mean-reverting process



* rate r itself is not traded but

+ Money market account M

$$dM_t = r_t M_t \, dt$$

+ Zero coupon bonds of various maturities T

$$\left(P_t(T) \right)_{0 \le t \le T}$$

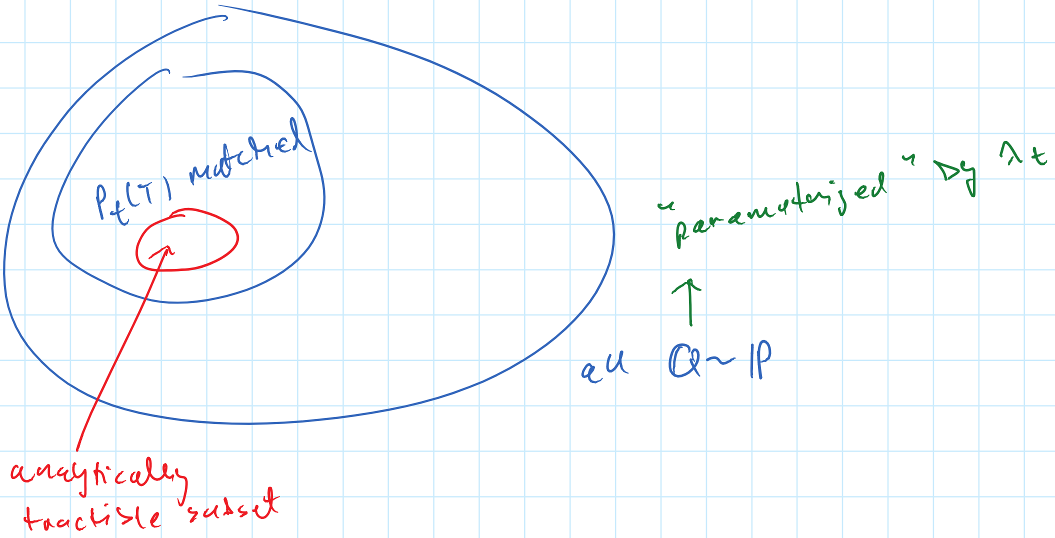
Bonds are contingent claims on the interest rate!

× From general theory of dynamic hedging / no arbitrage, $\exists \ \mathbb{Q} \sim \mathbb{P}$ s.t. relative prices of all traded assets (f) are $\mathbb{Q}$-mtg. Moreover,

$$\frac{f_t}{M_t} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{f_T}{M_T} \right]$$

and,

$$dr_t = (\mu_t - \underset{\substack{\uparrow \\ \text{market price of} \\ \text{risk}}}{\lambda_t} \sigma_t)\, dt + \sigma_t \, d\underset{\llcorner\ \mathbb{Q}\text{- B.mtn.}}{\hat{W}_t}$$



e.g. Vasicek :

$$dr_t = \kappa(\theta - r_t)\, dt + \sigma \, dW_t$$

$$= \left(\kappa(\theta - r_t) - \lambda_t \sigma \right) dt + \sigma \, d\hat{W}_t$$

what is the choice of λ_t s.t. $\mathbb{P}$ & $\mathbb{Q}$ models what is the choice of λ_t s.t. $\mathbb{P}$ & $\mathbb{Q}$ models are in the same class of models.

- $\lambda_t = \text{const.}$ → allows us to "change" θ
- $\lambda_t = a + b\, r_t$ → allows us to "change" θ, κ
- $\lambda_t = a_t + b_t\, r_t$

deterministic fn. of time

$$dr_t = \kappa_t (\theta_t - r_t)\, dt + \sigma\, d\hat{W}_t$$

just fn. of a_t, b_t, κ, θ.

$$\frac{P_t(T)}{M_t} = \mathbb{E}^{\mathbb{Q}}_t \left[\frac{\overset{\to 1}{P_T(T)}}{M_T} \right]$$

$$\Rightarrow \quad P_t(T) = \mathbb{E}^{\mathbb{Q}}_t \left[e^{-\int_t^T r_s\, ds} \right]$$

Q- B.motn

$$dr_t = \kappa(\theta_t - r_t)\,dt + \sigma\,d\hat{W}_t$$

$$P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,ds} \right]$$

① Find law of $\int_t^T r_s\,ds$

② PDE connection

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① $\int_t^T (\qquad)$

(*) $$\underline{r_T} - r_t = \kappa \int_t^T \theta_s\,ds - \kappa \underline{\int_t^T r_s\,ds} + \sigma \int_t^T d\hat{W}_s$$

need to find r_T in terms of $(W_s)_{t\le s\le T}$.

set $r_t = e^{-\kappa t} g_t$

↳ Find SDE for g_t.

$$\Rightarrow dr_t = d(e^{-\kappa t})\, g_t + e^{-\kappa t}\, dg_t + \overset{0}{d[\underline{e^{\kappa t}}, g_t]}$$

$$= -\kappa \underbrace{e^{-\kappa t} g_t}_{r_t}\, dt + e^{-\kappa t}\, dg_t$$

$\downarrow$

$$\kappa(\theta_t - r_t)\,dt + \sigma\,d\hat{W}_t$$

$$\Rightarrow \kappa\theta_t\,dt + \sigma\,d\hat{W}_t = e^{-\kappa t}\,dg_t$$

$$\boxed{dg_t = \kappa e^{\kappa t}\theta_t\,dt + \sigma e^{\kappa t}\,d\hat{W}_t}$$

(alternatively $g_t = e^{\kappa t} r_t \Rightarrow$)

$$\Rightarrow g_T - g_t = \kappa \int_t^T e^{\kappa s}\theta_s\,ds + \sigma \int_t^T e^{\kappa s}\,d\hat{W}_s$$

$$\Rightarrow e^{\kappa T} r_T - e^{\kappa t} r_t = \kappa \int_t^T e^{\kappa s}\theta_s\,ds + \sigma \int_t^T e^{\kappa s}\,d\hat{W}_s$$

$$\Rightarrow \boxed{r_T = e^{-\kappa(T-t)} r_t + \kappa \int_t^T e^{-\kappa(T-s)} \theta_s \, ds + \sigma \int_t^T e^{-\kappa(T-s)} d\hat{W}_s}$$

$$\left(\text{if } \theta_s = \text{const.} \quad \kappa \int_t^T e^{-\kappa(T-s)} \theta_s ds = (1 - e^{-\kappa(T-t)}) \theta . \right)$$

from (*) $\int_t^T r_s \, ds = \frac{r_t - r_T}{\kappa} + \int_t^T \theta_s \, ds + \frac{\sigma}{\kappa} \int_t^T d\hat{W}_s$

$$\Rightarrow \boxed{\int_t^T r_s ds = \frac{(1 - e^{-\kappa(T-t)})}{\kappa} r_t + \int_t^T (1 - e^{-\kappa(T-s)}) \theta_s \, ds + \frac{\sigma}{\kappa} \int_t^T (1 - e^{-\kappa(T-s)}) \, d\hat{W}_s}$$

$$\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T r_s \, ds \right] = \frac{1 - e^{-\kappa(T-t)}}{\kappa} r_t + \int_t^T (1 - e^{-\kappa(T-s)}) \theta_s \, ds$$

$$\mathbb{V}_t^{\mathbb{Q}}\left[\int_t^T r_s \, ds \right] = \frac{\sigma^2}{\kappa^2} \cdot \int_t^T (1 - e^{-\kappa(T-s)})^2 \, ds$$

moreover $\left. \int_t^T r_s ds \right|_{\mathcal{F}_t} \sim \mathcal{N}(\cdot , \cdot)$

Finally, $P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \right]$

$$= \exp\left\{ -\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T r_s ds \right] + \frac{1}{2} \mathbb{V}_t^{\mathbb{Q}}\left[\int_t^T r_s ds \right] \right\}$$

$$\boxed{P_t(T) = e^{A_t^\theta(T) - \underbrace{B_t(T)}_{\frac{1 - e^{-\kappa(T-t)}}{\kappa}} r_t}}$$

since $P = e^{\cdot + \cdot \, r_t}$ the model is called affine model.

$$A_t^\theta(T) = -\int_t^T (1 - e^{-\kappa(T-s)}) \theta_s \, ds + \frac{\sigma^2}{2\kappa^2} \int_t^T (1 - e^{-\kappa(T-s)})^2 \, ds$$

How to match bond prices?

$$\ln P^*_t(T) = -\int_t^T (1-e^{-\kappa(T-s)})\,\theta_s\,ds + \frac{\sigma^2}{2\kappa^2}\int_t^T (1-e^{-\kappa(T-s)})^2\,ds - \frac{1-e^{-\kappa(T-t)}}{\kappa}\,r_t$$

want to find θ_t s.t. above holds $\forall\, T \geq t$

$$\partial_T \ln P^*_t(T) = -\cancelto{0}{(1-e^{-\kappa(T-T)})}\,\theta_T - \int_t^T \kappa e^{-\kappa(T-s)}\,\theta_s\,ds + \frac{\sigma^2}{2\kappa^2}\cancelto{0}{(1-e^{-\kappa(T-T)})^2} + \frac{\sigma^2}{2\kappa^2}\int_t^T 2\kappa(1-e^{-\kappa(T-s)})\,e^{-\kappa(T-s)}\,ds - e^{-\kappa(T-t)}\,r_t$$

$$\partial_{TT} \ln P^*_t(T) = \kappa^2 e^{-\kappa(T-T)}\,\theta_T + \int_t^T \kappa^2 e^{-\kappa(T-s)}\,\theta_s\,ds + \frac{\sigma^2}{2\kappa^2}\int_t^T 2\kappa\left(-\kappa e^{-\kappa(T-s)} + 2\kappa e^{-2\kappa(T-s)}\right)ds + \kappa e^{-\kappa(T-t)}\,r_t$$

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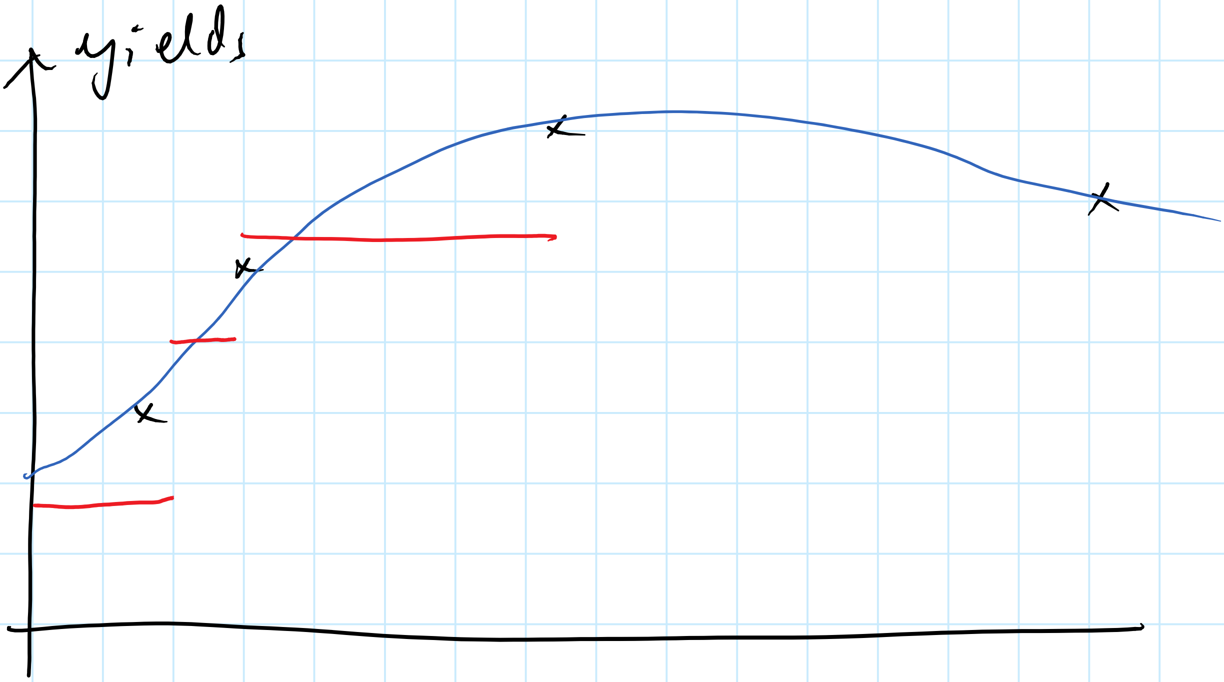

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PDE approach:

$$P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s\, ds} \right] = g(t, r_t)$$

$$dr_t = \kappa(\theta_t - r_t)\, dt + \sigma\, d\hat{W}_t$$

so, $g(\cdot,\cdot)$ satisfies

$$\begin{cases} (\partial_t + \mathcal{L})\, g = r\, g \\ g(T, r) = 1 \end{cases}$$

$$\mathcal{L} = \underbrace{\kappa(\theta_t - r)}_{}\, \partial_r + \underbrace{\tfrac{1}{2}\sigma^2}_{}\, \partial_{rr}$$

linear in r so it is called affine.

expect $g = e^{A_t - B_t\, r}$, $A_T = 0$, $B_T = 0$

deterministic fn. time (do not depend on r)

$$\partial_t g = (\dot{A} - \dot{B}\, r)\, g$$

$$\partial_r g = -B g, \quad \partial_{rr} g = B^2 g$$

$$\Rightarrow (\dot{A} - \dot{B}\, r) + \kappa(\theta_t - r)(-B) + \tfrac{1}{2}\sigma^2 B^2 = r$$

$$(\dot{A} - \kappa B \theta_t + \tfrac{1}{2}\sigma^2 B^2)$$

$$+ (-\dot{B} + \kappa B - 1)\, r = 0$$

must hold $\forall\, r, t \Rightarrow$

$$\begin{cases} \dot{A} - \kappa B \theta_t + \frac{1}{2}\sigma^2 B^2 = 0 & ① \\ \dot{B} - \kappa B + 1 = 0 & ② \end{cases}$$

$$② \Rightarrow \quad B = \frac{1 - e^{-\kappa(T-t)}}{\kappa}$$

$$① \Rightarrow \quad \underset{0 \ \nearrow}{A_T} - A_t - \int_t^T \left(1 - e^{-\kappa(T-s)}\right) \theta_s \, ds$$

$$+ \frac{1}{2}\frac{\sigma^2}{\kappa^2} \int_t^T \left(1 - e^{-\kappa(T-s)}\right)^2 ds$$

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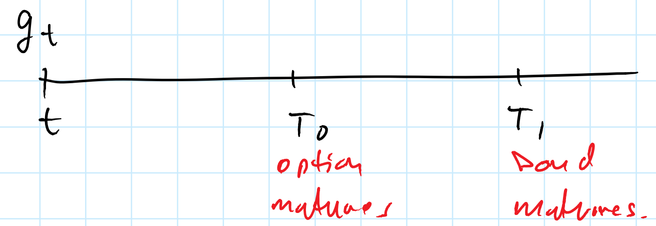
Bond Options

· call $\quad (P_t(T_1) - K)_+ \quad$ paid @ T_0



$$\frac{g_t}{M_t} = \mathbb{E}_t^{Q}\left[\frac{g_{T_0}}{M_{T_0}}\right]$$

$$\Rightarrow \quad g_t = \mathbb{E}_t^{Q}\left[e^{-\int_t^{T_0} r_s\,ds} \Big(\underbrace{P_{T_0}(T_1)}_{e^{A_{T_0}(T_1) - B_{T_0}(T_1)\,r_t}} - K\Big)_+ \right]$$

use T_0-bond as numeraire instead of M!

$$\frac{g_t}{P_t(T_0)} = \mathbb{E}_t^{Q_0}\left[\frac{g_{T_0}}{\underbrace{P_{T_0}(T_0)}_{1}}\right]$$

$$\Rightarrow \quad g_t = P_t(T_0)\,\mathbb{E}_t^{Q_0}\left[\left(P_{T_0}(T_1) - K\right)_+ \right]$$

recall that $\quad P_t(T_1) = e^{A_t - B_t r_t}$

and so,
$$\frac{dP_t(T_1)}{P_t(T_1)} = r_t\,dt - B_t(T_1)\,\sigma\,d\hat{W}_t$$

$$\big(\underbrace{\partial_r P_t(T_1)}_{-B\,.\,P}\big)\,\big(\sigma\,d\hat{W}_t\big)$$

$$dP_t(T_1) = \underbrace{(\partial_t + \mathcal{L})\,P}_{r\,P}\;dt + \underbrace{\partial_r P\;\sigma}_{-B\,P\,\sigma}\;d\hat{W}_t$$

$$\underbrace{\qquad\qquad}_{rP} \qquad\qquad \underbrace{\qquad\qquad}_{-BP\sigma}$$

$$\frac{dP_t(T_0)}{P_t(T_0)} = r_t\,dt - B_t(T_0)\,\sigma\,d\hat{W}_t$$

so $d\hat{W}^0_t = B_t(T_0)\sigma\,dt + d\hat{W}_t$

$\mathbb{Q}_0$ - B. mtg

so then, $$\frac{dP_t(T_1)}{P_t(T_1)} = (r_t + \sigma^2 B_t(T_0)\,B_t(T_1))\,dt - \sigma B_t(T_1)\,d\hat{W}^0_t$$

introduce: $$X_t = \frac{P_t(T_1)}{P_t(T_0)} \quad !$$

note $$X_{T_0} = \frac{P_{T_0}(T_1)}{P_{T_0}(T_0)} = P_{T_0}(T_1)$$

also X is a $\mathbb{Q}_0$-mtg.

$$\frac{dX_t}{X_t} = \underbrace{\sigma\,(-B_t(T_1) + B_t(T_0))}_{\Sigma_t}\,d\hat{W}^0_t$$

so $$X_T = X_t\, e^{-\frac{1}{2}\int_t^T \Sigma_s^2\,ds + \int_t^T \Sigma_s\,d\hat{W}^0_s}$$

and $$g = P_t(T_0)\,\mathbb{E}^{\hat{\mathbb{Q}}_0}\left[\,(X_{T_0} - K)_+\,\right]$$

$$\boxed{\begin{aligned} J_t &= P_t(T_0) \cdot \left\{ X_t \, \Phi(d_+) - K \, \Phi(d_-) \right\} \\ d_{\pm} &= \frac{\ln(X_t/K) \pm \frac{1}{2} \int_t^T \Sigma_s^2 \, ds}{\sqrt{\int_t^T \Sigma_s^2 \, ds}} \end{aligned}}$$