Wednesday, November 03, 2010
10:01 AM

1. Feynman-Kac & Girsanov's Thm

2. IR model

3. numeraire measures

Feynman-Kac Thm

$$f(t,s) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u,S_u)\,du} \; \varphi(S_T) \;\middle|\; S_t = S \right]$$

$$\frac{dS_t}{S_t} = \mu(t,S_t)\,dt + \sigma(t,S_t)\,d\hat{W}_t$$

$\hat{W}_t$ $\llcorner$ $\mathbb{Q}$-B.mtn

then f is the unique sol of

$$\begin{cases} \partial_t f + \mu(t,s)\, S\, \partial_s f + \frac{1}{2}\sigma^2(t,s)\, S^2\, \partial_{ss} f = r(t,s)\, f \\ f(T,s) = \varphi(s) \end{cases}$$

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recall that if X_t is our underlying risk

- $dX_t = \mu^x(t,X_t)\,dt + \sigma^x(t,X_t)\,dW_t$

  W_t $\llcorner$ $\mathbb{P}$-B. mtn.

- $\dfrac{dM_t}{M_t} = r(t,X_t)\,dt$ traded

- $\dfrac{df_t}{f_t} = \mu^f(t,X_t)\,dt + \sigma^f(t,X_t)\,dW_t$ traded

then dynamic hedging argument $\Rightarrow$ value of a claim $g_t = g(t,X_t)$ s.t. $g_T = \varphi(X_T)$ satisfies:

$$\begin{cases} \partial_t g + (\mu^x - \sigma^x \lambda)\, \partial_x g + \frac{1}{2}(\sigma^x)^2\, \partial_{xx} g = r\, g \\ g(T,x) = \varphi(x) \end{cases}$$

m.p.r.

$$\lambda = \frac{\mu^H - r}{\sigma^H}$$

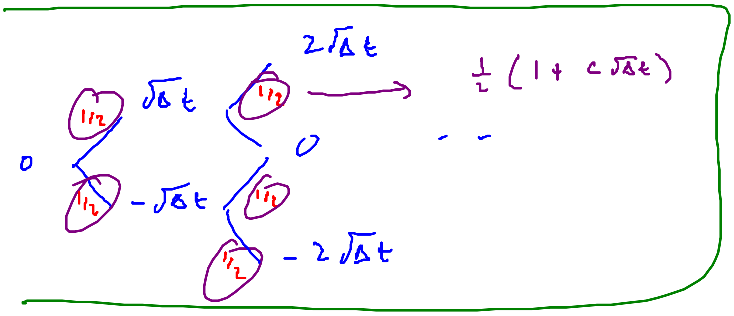
Feynman-Kac $\Rightarrow$

T

$$g(t,x) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u,X_u)\,du} \, \varphi(X_T) \,\middle|\, X_t = x \right]$$

and $$dX_t = \left(\mu^x(t,X_t) - \sigma^x(t,X_t)\,\lambda(t,X_t) \right) dt + \sigma^x(t,X_t)\, d\underbrace{\hat{W}_t}_{\mathbb{Q}\text{-B.mtn.}}$$

recall $$dX_t = \mu^x(t,X_t)\,dt + \sigma^x(t,X_t)\, d\underbrace{W_t}_{\mathbb{P}\text{-B.mtn}}$$



suggests that

$$\boxed{d\hat{W}_t = \lambda(t,X_t)\,dt + dW_t}$$

intuition is that $\hat{W}_t$ is a drift adjusted W_t.
(m.p.r.)

¿ how to relate $\mathbb{P}$ to $\mathbb{Q}$?

$$\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T = \exp\left\{ -\frac{1}{2}\int_0^T \lambda_u^2\,du - \int_0^T \lambda_u\,dW_u \right\}$$

is a Radon-Nikodym derivative.

$A \in \mathcal{F}_T$
↳ event measurable

$$\mathbb{Q}(A) = \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_A] = \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_A \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T \right]$$

$$\int_{\omega \in A} d\mathbb{Q}(\omega) = \int_{\omega \in A} \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T(\omega)\; d\mathbb{P}(\omega)$$

$$\mathbb{Q}(A \mid \mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_A \mid \mathcal{F}_t]$$

$$= \frac{\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_A \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T \Big| \mathcal{F}_t \right]}{\underbrace{\mathbb{E}^{\mathbb{P}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T \Big| \mathcal{F}_t \right]}_{\eta_t \text{ is } \mathbb{P}\text{-mtg.}}}$$

$$\begin{array}{lll} & S e^{\sigma\sqrt{\Delta t}} & \\ S & & \\ & S e^{-\sigma\sqrt{\Delta t}} & \end{array}$$

$$p = \frac{1}{2}\left(1 + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right)$$

$$q = \frac{1}{2}\left(1 + \frac{r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right)$$

$$\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)(x_1 = +1) = \frac{q}{p}$$

$$\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)(x_1 = -1) = \frac{1-q}{1-p}$$

$$\begin{array}{ccccc} \mathbb{P} & & \frac{d\mathbb{Q}}{d\mathbb{P}} & & \mathbb{Q} \\ \downarrow & & \downarrow & & \downarrow \\ p & \times & \frac{q}{p} & = & q \end{array}$$

$$(1-p)\cdot\left(\frac{1-q}{1-p}\right) = 1-q$$

Girsanov's Theorem:

$$\hat{W}_t = \int_0^t \lambda(t, X_u)\, du + W_t$$

is a $\mathbb{Q}$- B.mtn.

$$Z \underset{\mathbb{P}}{\sim} \mathcal{N}(0,1)$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\frac{1}{2}\lambda^2 - \lambda Z}$$ here λ = const.

¿ what is the $\mathbb{Q}$- distribution of Z?

→ compute char. fn / m.g.f.

$$\mathbb{E}^{\mathbb{Q}}[e^{uZ}] = \mathbb{E}^{\mathbb{P}}\left[e^{uZ}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[e^{uZ} e^{-\frac{1}{2}\lambda^2 - \lambda Z}\right]$$

$$= e^{-\frac{1}{2}\lambda^2}\mathbb{E}^{\mathbb{P}}\left[e^{(u-\lambda)Z}\right]$$

$$= e^{-\frac{1}{2}\lambda^2} e^{\frac{1}{2}(u-\lambda)^2}$$

$$= e^{\frac{1}{2}u^2 - u\lambda}$$

$$\therefore \quad Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(-\lambda; 1)$$

$$Z \underset{\mathbb{P}}{\sim} \mathcal{N}(0; 1)$$

set $\hat{Z} = \lambda + Z$

then $\hat{Z} \sim \mathcal{N}(0; 1)$

$\mathbb{Q}$

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Vasicek Models:

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\, dW_t$$

$\mathbb{P}$ - B.mtn

$$\rightarrow \quad r_t = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma \int_0^t e^{-\kappa(t-u)}\, dW_u$$

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u\, du}\, \varphi(r_T) \mid \mathcal{F}_t \right]$$

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\left(d\hat{W}_t - \lambda(t, r_t)\,dt\right)$$

$\mathbb{Q}$ - B.mtn.

$$= \left(\kappa\theta - \kappa r_t - \sigma\lambda(t, r_t)\right)dt + \sigma\, d\hat{W}_t$$

$a + b r_t$

$$= \hat{\kappa}(\hat{\theta} - r_t)\,dt + \sigma\, d\hat{W}_t$$

$$\hat{\kappa} = \kappa + b\sigma$$
$$\hat{\theta} = \frac{\kappa\theta - a\sigma}{\kappa + b\sigma}$$

so $$r_t = r_0 e^{-\hat{\kappa} t} + \theta(1 - e^{-\hat{\kappa} t}) + \sigma \int_0^t e^{-\hat{\kappa}(t-u)}\, d\hat{W}_u$$

$$\mathbb{E}^{\mathbb{Q}}[r_t] = r_0 e^{-\hat{\kappa} t} + \theta(1 - e^{-\hat{\kappa} t})$$

$$\mathbb{V}^{\mathbb{Q}}[r_t] = \mathbb{V}^{\mathbb{Q}}\left[\sigma \int_0^t e^{-\hat{\kappa}(t-u)}\, d\hat{W}_u\right]$$

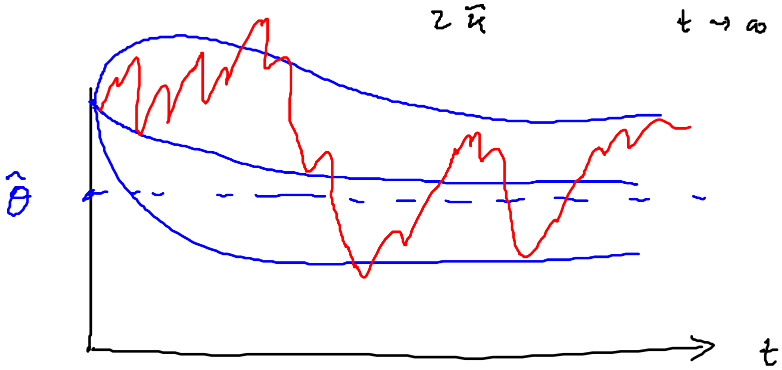
$$= \sigma^2\, \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^t e^{-\hat{\kappa}(t-u)}\, d\hat{W}_u\right)^2\right]$$

Ito's isometry

$$= \sigma^2\, \mathbb{E}^{\mathbb{Q}}\left[\int_0^t e^{-2\hat{\kappa}(t-u)}\, du\right]$$

$$= \sigma^2 \frac{1 - e^{-2\hat{\kappa} t}}{\quad} \longrightarrow \frac{\sigma^2}{\quad}$$

$z\ \bar{x}$ $\qquad$ $t \to \infty \quad z\ \bar{x}$



value a bond ...

payoff $= 1 \;\; \forall \, r_T$

$$P_t(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u \, du} \,\middle|\, \mathcal{F}_t \right]$$

$$dr_t = \hat{\kappa}(\hat{\theta} - r_t)\,dt + \sigma \, d\hat{W}_t$$

$$r_T - r_t = \hat{\kappa}\hat{\theta}(T-t) - \hat{\kappa}\int_t^T r_u \, du + \sigma \int_t^T d\hat{W}_u$$

$$\Rightarrow \int_t^T r_u \, du = \hat{\theta}(T-t) - \frac{1}{\hat{\kappa}}(r_T - r_t) + \frac{\sigma}{\hat{\kappa}}\int_t^T d\hat{W}_u$$

$$r_T = r_t e^{-\hat{\kappa}(T-t)} + \hat{\theta}\left(1 - e^{-\hat{\kappa}(T-t)}\right) + \sigma \int_t^T e^{-\kappa(T-u)} \, d\tilde{W}_u$$

$$\int_t^T r_u \, du = \underbrace{\hat{\theta}\left[(T-t) - \frac{1 - e^{-\hat{\kappa}(T-t)}}{\hat{\kappa}}\right] + \frac{1 - e^{-\hat{\kappa}(T-t)}}{\hat{\kappa}} r_t}_{m} + \frac{\sigma}{\hat{\kappa}}\int_t^T \left(1 - e^{-\hat{\kappa}(T-u)}\right) d\hat{W}_u$$

$$\sim \mathcal{N}(m \, ; \, v^2)$$

$$v^2 = \frac{\sigma^2}{\hat{\kappa}^2}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \left(1 - e^{-\hat{\kappa}(T-u)}\right)^2 du\right]$$

$$= \frac{\sigma^2}{\hat{\kappa}^2}\left((T-t) - 2\,\frac{1 - e^{-\hat{\kappa}(T-t)}}{\hat{\kappa}} + \frac{1 - e^{-2\hat{\kappa}(T-t)}}{2\hat{\kappa}} \right)$$

$$\therefore \; P_t(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u \, du} \,\middle|\, \mathcal{F}_t \right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[e^{-(m + v Z)} \mid \mathcal{F}_t \right)$$

$$Z \underset{\mathbb{Q}}{\sim} N(0,1)$$

$$= e^{-m + \frac{1}{2}v^2}$$

$$= \exp\left\{ A_t(T) - B_t(T)\, r_t \right\}$$

only source of uncertainty.

$$\frac{1 - e^{-\hat{\kappa}(T-t)}}{\hat{\kappa}}$$

Vasicek is an Affine model

recall that

$$\begin{cases} \partial_t P + \hat{\kappa}(\hat{\theta} - r)\,\partial_r P + \frac{1}{2}\sigma^2 \partial_{rr} P = r P \\ P(T, r) = 1 \end{cases}$$

assume that $P(t, r; T) = \exp\{ \underbrace{A_t(T) - B_t(T)}\, r \}$ fns only of t not r.
(affine form)

$$\partial_t P = (\dot{A} - \dot{B}\, r)\, P$$

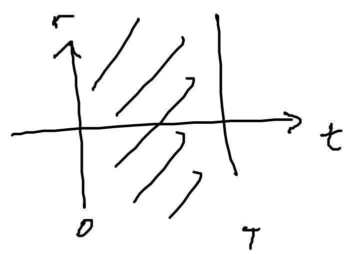
$$\partial_r P = -B\, P$$

$$\partial_{rr} P = B^2\, P$$

$$(\dot{A} - \dot{B}\, r) + \hat{\kappa}(\hat{\theta} - r)(-B) + \frac{1}{2}\sigma^2 B^2 = r$$

$$\left(\dot{A} - \hat{\kappa}\hat{\theta} B + \frac{1}{2}\sigma^2 B^2\right)$$

$$+ \left(-\dot{B} + \hat{\kappa} B - 1\right) r = 0$$



must hold $\forall$ t & r

$$\Leftrightarrow \text{①} \quad \dot{A} - \hat{\kappa}\hat{\theta} B + \frac{1}{2}\sigma^2 B^2 = 0$$

$$\& \text{②} \quad -\dot{B} + \hat{\kappa} B - 1 = 0$$

Riccatti Equations ODEs.

b.c. $A_T(T) - B_T(T)\, r = 0 \quad \forall r$

$$\Leftrightarrow A_T(T) = 0 \;\&\; B_T(T) = 0$$

$$B_t(T) = \frac{1 - e^{-\hat{\kappa}(T-t)}}{\hat{\kappa}}$$

② $\Rightarrow \quad \dot{B} = \hat{\kappa} B - 1$

$$\Rightarrow \quad \frac{\dot{B}}{\hat{\kappa} B - 1} = 1 \quad \Rightarrow \quad \frac{1}{\hat{\kappa}} d\left(\ln(\hat{\kappa} B - 1)\right) = 1$$

$$\Rightarrow \quad \frac{1}{\hat{\kappa}}\left[\ln(\hat{\kappa}\, B_T(T) - 1) - \ln(\hat{\kappa}\, B_t(T) - 1)\right] = T - t$$

$B_T(T) \to 0$

$$\Rightarrow \quad -\ln\left(1 - \hat{\kappa}\, B_t(T)\right) = \hat{\kappa}(T-t)$$

$$\Rightarrow \quad 1 - \hat{\kappa}\, B_t(T) = e^{-\hat{\kappa}(T-t)}$$

$$\Rightarrow \quad B_t(T) = \frac{1 - e^{-\hat{\kappa}(T-t)}}{\hat{\kappa}}$$

$$\dot{A} - \hat{\kappa}\,\hat{\theta}\, B + \frac{1}{2}\sigma^2 B^2 = 0$$

$$\Rightarrow \quad A_T - A_t - \hat{\kappa}\,\hat{\theta} \int_t^T B_u\, du + \frac{1}{2}\sigma^2 \int_t^T B_u^2\, du = 0$$

$A_T \to 0$

$$\Rightarrow \quad A_t = -\hat{\kappa}\,\hat{\theta} \int_t^T B_u\, du + \frac{1}{2}\sigma^2 \int_t^T B_u^2\, du$$

$$= \ldots$$