Wednesday, October 20, 2010
10:23 AM

1 - 2:30 T Oct 26

- 3 important results
- ? what's bothering you?

Wednesday, October 20, 2010
10:27 AM

Continuous Time dynamic hedging

key tools are Ito processes & Ito's lemma
$\rightarrow$ Feynmann-Kac Thm.

- underlying risk in economy X_t is an Ito process

$$\frac{dX_t}{X_t} = \underbrace{\mu^X(t, X_t)}_{X\text{-drift}} dt + \underbrace{\sigma^X(t, X_t)}_{X\text{-vol}} dW_t$$

$\llcorner$ $\mathbb{P}$-B. mtn

$\in \mathcal{F}_t$

- money market account: M_t

$$\frac{dM_t}{M_t} = \underbrace{r(t, X_t)}_{\text{risk-free rate} \;\in \mathcal{F}_t} dt$$

- risky-asset: $f_t = f(t, X_t)$

$$\frac{df_t}{f_t} = \underbrace{\mu^f(t, X_t)}_{f\text{-drift}} dt + \underbrace{\sigma^f(t, X_t)}_{f\text{-vol}} dW_t$$

$\in \mathcal{F}_t$

Want to value a new claim $g_t = g(t, X_t)$

(Euro) pays $\varphi(X_T)$ @ T

---

trade α_t units of f_t
β_t units of M_t
-1 unit of g_t

$$V_t = \alpha_t f_t + \beta_t M_t - g_t$$

- $V_0 = 0$

$$dV_t = d(\alpha_t f_t) + d(\beta_t M_t) - dg_t$$

$$= \alpha_t\, df_t + \beta_t\, dM_t - dg_t$$

$$\underbrace{\boxed{\begin{array}{l} + f_t\, d\alpha_t + M_t\, d\beta_t \\ + d[\alpha, f]_t + d[\beta, M]_t \end{array}}}_{=0}$$

self-financing condition

$$dV_t = \alpha_t\, df_t + \beta_t\, dM_t - dg_t$$

self-financing condition

$$= \alpha_t f_t (\mu_t^f\, dt + \sigma_t^f\, dW_t)$$

$$+ \beta_t\, r_t M_t\, dt$$

$$- g_t (\mu_t^g\, dt + \sigma_t^g\, dW_t)$$

recall: Ito's lemma

$$dg_t = \underbrace{\left(\partial_t g + \mu_t^x\, x_t\, \partial_x g_t + \tfrac{1}{2}(\sigma_t^x)^2 x_t^2\, \partial_{xx} g_t\right)}_{\mu_t^g\, g_t} dt$$

$$+ \underbrace{\sigma_t^x\, x_t\, \partial_x g_t}_{\sigma_t^g\, g_t}\, dW_t$$

$$\Rightarrow \frac{dg_t}{g_t} = \mu_t^g\, dt + \sigma_t^g\, dW_t$$

choose α_t s.t. $(\quad)\, dW \to 0$

$$\Rightarrow \boxed{\alpha_t = \frac{g_t\, \sigma_t^g}{f_t\, \sigma_t^f}}$$

leads to zero instantaneous risk!

$$dV_t = \underbrace{\left(\alpha_t f_t \mu_t^f + \beta_t M_t r_t - g_t \mu_t^g\right)}_{\in \mathcal{F}_t} dt + 0 \cdot dW_t$$

$\hookrightarrow$ must be 0 to avoid arbitrage.

$V_0 = 0$ & $dV_t = 0 \quad \forall t$

$\Rightarrow V_t = 0 \quad \forall t \Rightarrow \alpha_t f_t + \beta_t M_t - g_t = 0$

$\Rightarrow \beta_t M_t = g_t - \alpha_t f_t$

then: $\left(\frac{g_t \sigma_t^g}{f_t \sigma_t^f}\right) f_t \mu_t^f + \left(g_t - \frac{g_t \sigma_t^g}{f_t \sigma_t^f} \cdot f_t \right) r_t - g_t \mu_t^g = 0$

$$\Rightarrow \boxed{\frac{\mu_t^f - r_t}{\sigma_t^f} = \frac{\mu_t^g - r_t}{\sigma_t^g}} = \lambda(t, X_t)$$

market's price of risk

are Sharpe Ratios!

$\mu_t^g - r_t = \lambda_t \sigma_t^g$

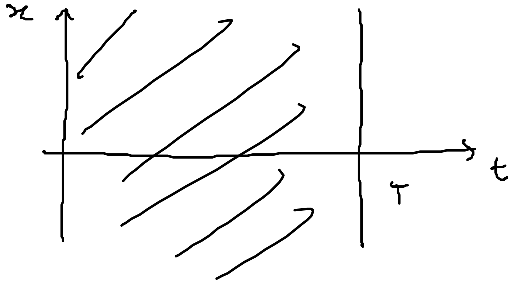
$\Rightarrow \mu_t^g - \lambda_t \sigma_t^g = r_t$

$$\partial_t g(t,X_t) + \mu^x(t,X_t) X_t \partial_x g(t,X_t) + \tfrac{1}{2}(\sigma^x(t,X_t) X_t)^2 \partial_{xx} g(t,X_t)$$
$$- \sigma(t,X_t)\lambda(t,X_t) X_t \partial_x g(t,X_t) = r_t g(t,X_t)$$

$$\Rightarrow \boxed{\begin{aligned} &\partial_t g(t,x) + \left(\mu^x(t,x) - \sigma(t,x)\lambda(t,x)\right) x \partial_x g(t,x) \\ &\quad + \tfrac{1}{2}\left(\sigma^x(t,x)\, x\right)^2 \partial_{xx} g(t,x) = r(t,x)\, g(t,x) \end{aligned}}$$

$\uparrow$

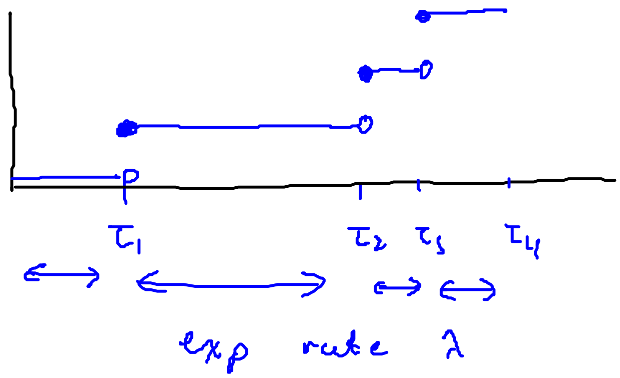
B-S
PDE



$g(T, x) = \Phi(x)$

Wednesday, October 20, 2010
12:08 PM

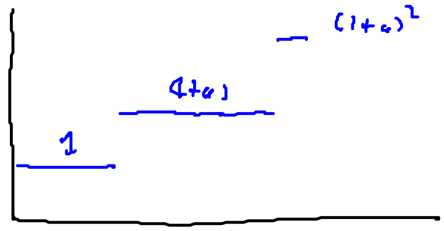
N_t - Poisson process : intensity is λ



$$\mathbb{P}(N_{t+\Delta t} - N_t = 0) = 1 - \lambda \Delta t + \cdots$$
$$\mathbb{P}(N_{t+\Delta t} - N_t = 1) = \lambda \Delta t + \cdots$$
$$\mathbb{P}(N_{t+\Delta t} - N_t \geq 2) = \cdots$$

* risky asset: $\frac{dF_t}{F_{t-}} = a\, dN_t$, $F_0 = 1$, $a > 0$

$$F_{t-} \longrightarrow F_{t-} + a F_{t-} = (1+a) F_{t-}$$



* $\frac{dM_t}{M_t} = r\, dt$

$G : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$.

* $g_t = G(t, F_t)$ $\qquad$ $g_t = G(t, F_t)$

$\longrightarrow$

$$dg_t = \partial_t g\, dt + \big(g(t, F_{t-}(1+a)) - g(t, F_{t-}) \big)\, dN_t$$

$\hookrightarrow \Delta g_t$

Wednesday, October 20, 2010
1:00 PM

$$\left.\begin{array}{llll} \alpha_t & - \text{units} & \text{of} & F_t \\ \beta_t & - \;\;" & " & M_t \\ -1 & - \;\;" & " & g_t \end{array}\right\} \qquad V_0 = 0$$

$$V_t = \alpha_t F_t + \beta_t M_t - g_t$$

$$dV_t = \alpha_t \, df_t + \beta_t \, dM_t - dg_t$$

self-financing.

$$= \alpha_t \, a F_t \, dN_t + \beta_t M_t r \, dt$$

$$- (\partial_t g \, dt + \Delta g \, dN_t)$$

choose $\alpha_t = \dfrac{\Delta g}{a F_t}$

$$\Rightarrow \quad dV_t = (r \beta_t M_t - \partial_t g) \, dt$$

since $dV_t \in \mathcal{F}_t \Rightarrow$ to avoid arb.

$$(\cdot) = 0$$

$$\Rightarrow \quad r \beta_t M_t - \partial_t g = 0 .$$

Further since $V_0 = 0$ & $dV_t = 0 \Rightarrow \forall t \;\; V_t = 0$

$$\Rightarrow \quad \alpha_t F_t + \beta_t M_t - g_t = 0$$

$$\Rightarrow \quad \beta_t M_t = g_t - \alpha_t F_t$$

$$\Rightarrow \quad r (g_t - \tfrac{1}{a} \Delta g) - \partial_t g = 0$$

$$\Rightarrow \quad \boxed{\partial_t g + \frac{r}{a} \Delta g = r g}$$

$$g(T, f) = Q(f).$$