Wednesday, October 06, 2010
9:58 AM

Ho - Lee

$$r_n = r_{n-1} + \sigma\sqrt{\Delta t}\; x_n + \theta_{n-1}\,\Delta t$$

avoids -ve r.

$$\hookrightarrow \quad r_n = r_{n-1} \exp\left\{ \sigma\sqrt{\Delta t}\; x_n + \theta_{n-1}\,\Delta t \right\}$$

BDT

Black, Derman, Toy

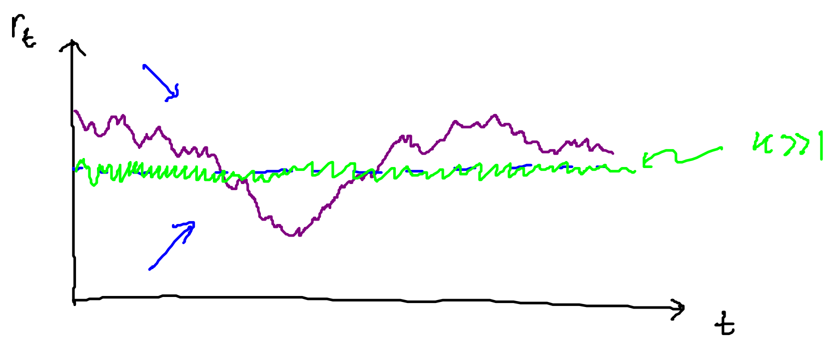
$$r_T \xrightarrow[\Delta t \downarrow 0]{d} r_0 \exp\left\{ \sigma\sqrt{T}\; Z + \int_0^T \theta_s\, ds \right\}$$

$$Z \underset{\mathbb{Q}}{\sim} N(0,1)$$

$$P_0(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s\, ds} \right]$$

$$e^X + e^Y \overset{d}{=} ? \longleftarrow \int_0^T r_s\, ds$$

→ mean-reversion ...



Vasicek - model

rate of m-r

vol

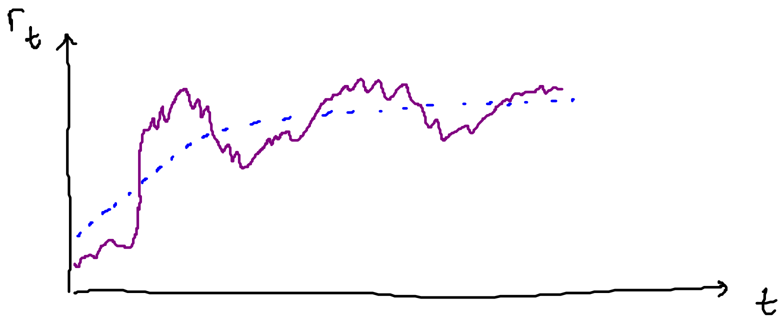
$$r_n - r_{n-1} = \kappa\,(\theta_{n-1} - r_{n-1})\,\Delta t + \sigma\sqrt{\Delta t}\; x_n$$

level of m-r

Bernoulli

$\mathbb{Q}(x_k = \pm 1) = \frac{1}{2}$

$$\kappa > 0, \quad \theta > 0, \quad \sigma > 0$$



$$r_n = \kappa\,\theta\,\Delta t + (1-\kappa\Delta t)\, r_{n-1} + \sigma\sqrt{\Delta t}\, x_n$$

$$= a + b\, r_{n-1} + \underbrace{\sigma\sqrt{\Delta t}\, x_n}_{\varepsilon_n}$$

this is an AR(1) model

$$r_n = a + b(a + b\, r_{n-2} + \varepsilon_{n-1}) + \varepsilon_n$$

$$= a(1+b) + b^2 r_{n-2} + \varepsilon_n + b\,\varepsilon_{n-1}$$

$$= a(1+b) + b^2(a + b\, r_{n-3} + \varepsilon_{n-2}) + \varepsilon_n + b\,\varepsilon_{n-1}$$

$$= a(1+b+b^2) + b^3 r_{n-3} + \varepsilon_n + b\,\varepsilon_{n-1} + b^2 \varepsilon_{n-2}$$

$$= \ldots$$

$$= \underbrace{a\sum_{m=0}^{n-1} b^m}_{A} + \underbrace{b^n}_{B} r_0 + \underbrace{\sum_{m=0}^{n-1} b^m \varepsilon_{n-m}}_{X}$$

$\Delta t = T/n$, $n \to +\infty$

$$B = (1-\kappa\Delta t)^n = (1-\kappa\Delta t)^{T/\Delta t} \xrightarrow[\Delta t \downarrow 0]{} e^{-\kappa T}$$

$$A = a\sum_{m=0}^{n-1} b^m = a\,\frac{1-b^n}{1-b}$$

$$= \kappa\,\theta\,\Delta t\; \frac{1-(1-\kappa\Delta t)^n}{}$$

$$1-(1-\kappa\Delta t)$$

$$= \theta \cdot \left(1-(1-\kappa\Delta t)^{n}\right) \xrightarrow[\Delta t \downarrow 0]{} \theta\left(1-e^{-\kappa T}\right)$$

$$\mathbb{E}^{\mathbb{Q}}[X] = 0$$

$$\mathbb{V}^{\mathbb{Q}}[X] = \mathbb{V}^{\mathbb{Q}}\left[\sum_{m=0}^{n-1} b^{m}\,\sigma\sqrt{\Delta t}\; x_{n-m}\right]$$

$$= \sum_{m=0}^{n-1} b^{2m}\,\sigma^{2}\Delta t\; \underbrace{\mathbb{V}^{\mathbb{Q}}[x_{n-m}]}_{\to 1}$$

$$= \sigma^{2}\Delta t\,\frac{1-b^{2n}}{1-b^{2}}$$

$$= \sigma^{2}\Delta t\,\frac{1-(1-\kappa\Delta t)^{2T/\Delta t}}{\left(1-(1-\kappa\Delta t)^{2}\right)}$$

$$\hookrightarrow 1-(1+\kappa^{2}\Delta t^{2}-2\kappa\Delta t)$$

$$= 2\kappa\Delta t-\kappa^{2}\Delta t^{2}$$

$$\to \frac{\sigma^{2}}{2\kappa}\left(1-e^{-2\kappa T}\right) = \hat{\sigma}_T^{2}$$

$$r_T \overset{d}{=} \theta(1-e^{-\kappa T}) + e^{-\kappa T} r_0 + \hat{\sigma}_T Z$$

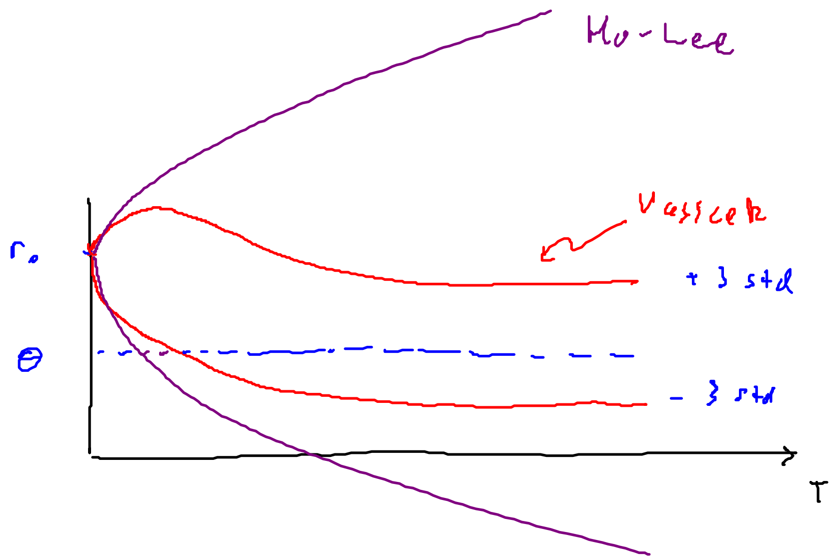
$$Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

$$\hat{\sigma}_T^{2} = \frac{\sigma^{2}}{2\kappa}\left(1-e^{-2\kappa T}\right)$$

$$\underset{T\ll 1}{\sim} \frac{\sigma^{2}}{2\kappa}\left(1-(1-2\kappa T)\right)$$

$$= \sigma^{2} T$$

$$\underset{T\gg 1}{\sim} \quad \frac{\sigma^2}{2\kappa}$$



now we need $I_T = \int_0^T r_s \, ds$

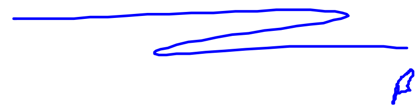
$$I_T = \lim_{\Delta t \downarrow 0} \sum_{n=1}^{N} r_{n-1} \, \Delta t$$

recall $r_n - r_{n-1} = \kappa(\theta - r_{n-1})\Delta t + \sigma\sqrt{\Delta t}\; x_n$

$$\underbrace{\sum_{n=1}^{N} (r_n - r_{n-1})}_{r_N - r_0} = \kappa\,\theta \underbrace{\Delta t \cdot N}_{T} - \kappa \underbrace{\sum_{n=1}^{N} r_{n-1}\,\Delta t}_{I_N} + \sigma\sqrt{\Delta t} \sum_{n=1}^{N} x_n$$

$$\Rightarrow I_N = \frac{1}{\kappa}\left[r_0 + \kappa\,\theta\, T - r_N + \underbrace{\sigma\sqrt{\Delta t} \sum_{n=1}^{N} x_n}_{\alpha} \right]$$

$$a \sum_{m=0}^{N-1} b^m + b^N r_0 + \sum_{m=0}^{N-1} b^m \, x_{N-m}\, \sigma\sqrt{\Delta t}$$



$$\alpha - \beta = \sigma\sqrt{\Delta t}\,\Big(x_1 + x_2 + \cdots + x_N$$

$$- \big(x_N + x_{N-1} b + x_{N-2} b^2 + \cdots + b^{N-2} x_2 + b^{N-1} x_1\big)\Big)$$

$$= \sigma\sqrt{\Delta t} \sum_{m=1}^{N} x_m \left(1 - b^{N-m}\right)$$

$$\mathbb{V}^{\mathbb{Q}}(\kappa I_N) = \sigma^2 \Delta t \sum_{m=1}^{N} \left(1 - b^{N-m}\right)^2$$

$$= \sigma^2 \Delta t \sum_{m=1}^{N} \left(1 - 2 b^{N-m} + b^{2(N-m)}\right)$$

$$= \sigma^2 \Delta t \left(N - 2\,\frac{1-b^N}{1-b} + \frac{1-b^{2N}}{1-b^2} \right)$$

$b = 1 - \kappa \Delta t$

$$\xrightarrow[\Delta t \downarrow 0]{} \sigma^2 \left[T - 2\left(1 - e^{-\kappa T}\right) + \frac{\left(1 - e^{-2\kappa T}\right)}{2\kappa} \right]$$

$$\mathbb{E}^{\mathbb{Q}}[\kappa I_N] = r_0 + \kappa\theta T - \left(\theta\left(1 - e^{-\kappa T}\right) + e^{-\kappa T} r_0\right)$$

$$= \kappa\theta T + (r_0 - \theta)\left(1 - e^{-\kappa T}\right)$$

$$P_0(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s\, ds} \right]$$

$$= \exp\left\{ -\mathbb{E}^{\mathbb{Q}}[I_T] + \tfrac{1}{2}\mathbb{V}^{\mathbb{Q}}[I_T] \right\}$$

$$= \exp\left\{ A_0(T) - B_0(T)\, r_0 \right\}$$

Vasicek is an affine model

and so is Ho-Lee!

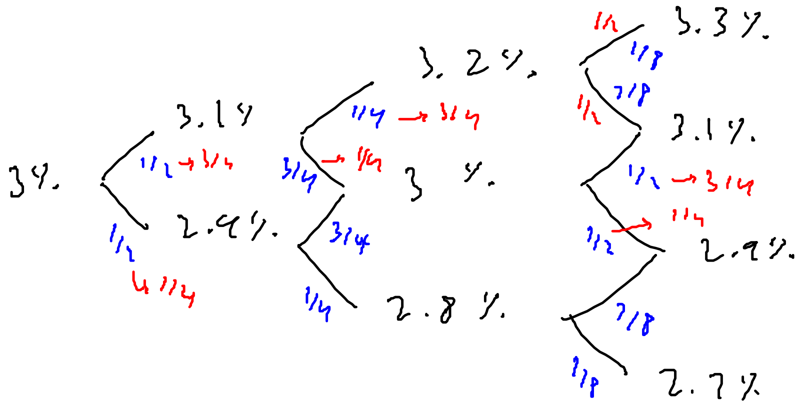
but not BDT

$$r_n - r_{n-1} = \underline{\underline{\kappa(\theta - r_{n-1})\Delta t}} + \sigma\sqrt{\Delta t}\; x_n$$

$$\mathbb{Q}(x_n = \pm 1) = \tfrac{1}{2}$$

$$r_n - r_{n-1} = \bar{\kappa}(\bar{\theta} - r_{n-1})\Delta t + \bar{\sigma}\sqrt{\Delta t}\; x_n$$

$$\mathbb{P}(x_n = \pm 1) = \tfrac{1}{2}$$



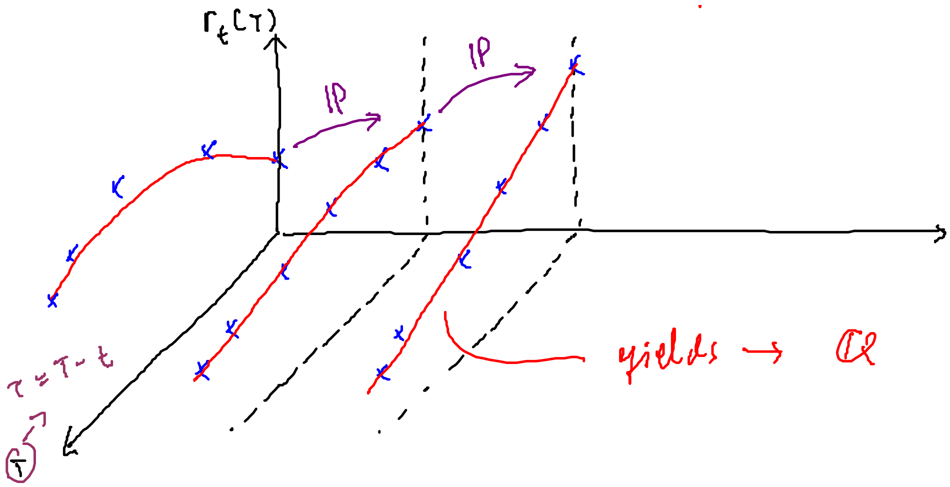
<u>term structure dynamics</u>

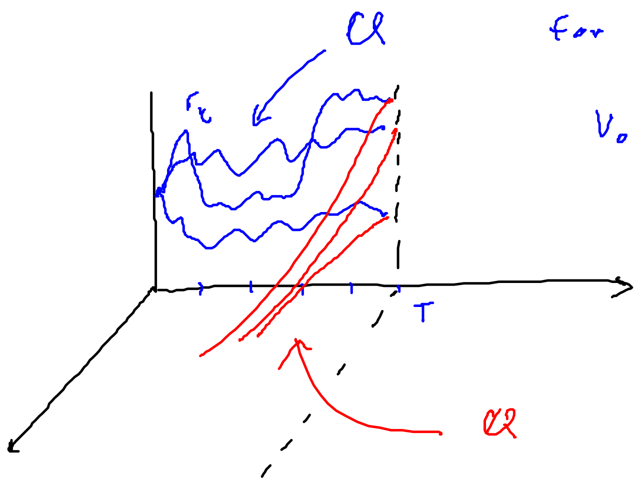
$$P_t(T) = e^{A_t(T) - r_t B_t(T)}$$

$$= e^{-r_t(T)\,(T-t)}$$

↳ yield

$r_t = \lim_{T \downarrow t} r_t(T)$

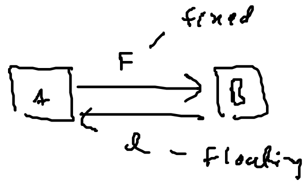


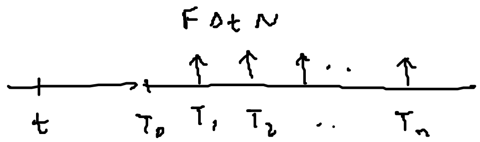


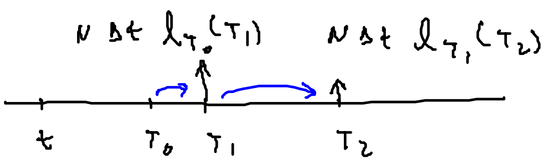
for derivative valuation.

$$V_0 = \mathbb{E}^{Q}\left[e^{-\int_0^T r_s ds} \left(P_T(T_1) - \alpha P_T(T_2) \right)_+ \right]$$

$$\frac{X_T - X_t}{X_t} = e^{-y(T-t)}$$







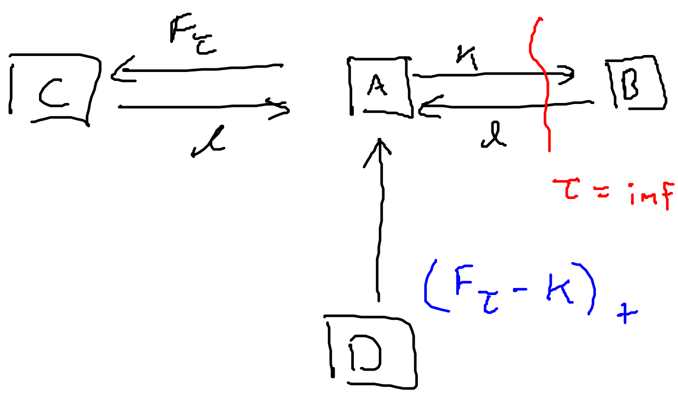
$$l_{T_{k-1}}(T_k) = \frac{1}{\Delta T}\left(\frac{1}{P_{T_{k-1}}(T_k)} - 1 \right)$$

$$V_t^{fl} = \left(P_t(T_0) - P_t(T_n) \right) N$$

$$V_t^{fix} = F \, \Delta t \, N \sum_{m=1}^{n} P_t(T_m)$$

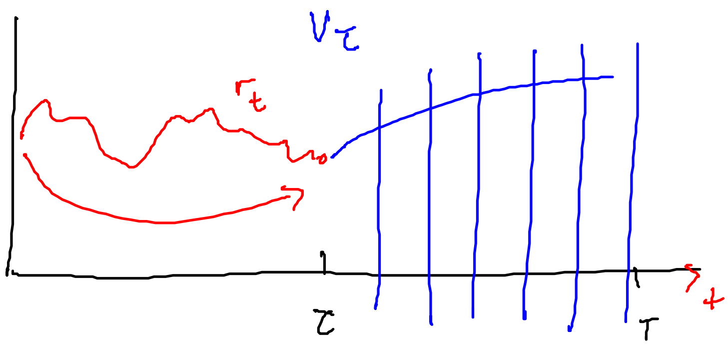
swap-rate at day t

$$\boxed{F_t = \frac{P_t(T_0) - P_t(T_n)}{\Delta t \sum_{m=1}^{n} P_t(T_m)}}$$



$\tau = \inf\{ t : N_t = 1 \}$ Poisson process

$N \Delta t \times$ $(F_\tau - K)_+$ $(F_\tau - K)_+$ $(F_\tau - K)_+$



$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{\tau} r_s\, ds} V_{\tau}(r_{\tau}) \right]$$

$$\mathbb{C}^{\mathbb{Q}}\left[\int_0^{\tau} r_s\, ds ,\ r_{\tau} \right] = ?$$