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i) $\left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right)_T = \frac{A_t / A_0}{M_t / M_0}$

ii) $dW_t^A = -\sigma_t^A \, dt + dW_t$

$dB_t^A = -\rho\,\sigma_t^A \, dt + dB_t \qquad d[B,W]_t = \rho\, dt$

if $\frac{dA_t}{A_t} = r_t\, dt + \sigma_t^A \, dW_t$

$\mathbb{Q}$- B.mtn

iii) $\frac{V_t}{A_t} = \mathbb{E}^{\mathbb{Q}^A}\left[\frac{V_T}{A_T} \,\middle|\, \mathcal{F}_t \right]$

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e.g. put (T) on a bond (u)

$$\frac{V_t}{P_t(T)} = \mathbb{E}^{\mathbb{Q}^T}\left[\frac{(K - P_T(u))_+}{\underbrace{P_T(T)}_{\to 1}} \,\middle|\, \mathcal{F}_t \right]$$

$$= \mathbb{E}^{\mathbb{Q}^T}\left[\left(K - \underbrace{\frac{P_T(u)}{P_T(T)}}_{X_T} \right)_+ \,\middle|\, \mathcal{F}_t \right]$$

X_T, $X_t = \frac{P_t(u)}{P_t(T)}$ is a $\mathbb{Q}^T$-mtg

recall that $P_t(T) = e^{A_t(T) - B_t(T)\, r_t}$

$dr_t = \kappa(\theta - r_t)\, dt + \sigma\, dW_t$

$$\frac{dP_t(T)}{P_t(T)} = \underbrace{r_t}_{\text{b/c } P_t(T) \text{ is traded}} dt - \sigma B_t(T)\, dW_t \qquad \frac{\partial_r P_t(T)}{P_t(T)}\sigma$$

$$\frac{dX_t}{X_t} = (\quad)\, dt - \sigma B_t(u)\, dW_t + \sigma B_t(T)\, dW_t$$

$$\begin{pmatrix}\text{convexity}\\ \text{correction}\end{pmatrix} dt + \frac{dP_t(u)}{P_t(u)} - \frac{dP_t(T)}{P_t(T)}$$

$$\Rightarrow \frac{dX_t}{X_t} = \sigma \left[B_t(T) - B_t(U) \right) dW_t^T$$

$$\left[\text{recall if } \frac{dY_t}{Y_t} = \sigma_t \, dW_t \right.$$

then

$$\left. Y_t = Y_0 \exp\left\{ -\frac{1}{2} \int_0^t \sigma_s^2 \, ds + \int_0^t \sigma_s \, dW_s \right\} \right]$$

$$\therefore \quad X_T \overset{d}{=} X_t \, e^{-\frac{1}{2} \bar{\sigma}^2 (T-t) + \bar{\sigma} \sqrt{T-t} \, Z}$$

$$Z \underset{\mathbb{Q}^T}{\sim} \mathcal{N}(0,1)$$

$$\bar{\sigma}^2 = \frac{1}{T-t} \int_t^T \sigma^2 \left(B_s(T) - B_s(U) \right)^2 ds$$

$$\text{so} \quad V_t = P_t(T) \, \mathbb{E}^{\mathbb{Q}^T}\left[(K - X_T)_+ \mid \mathcal{F}_t \right]$$

$$= P_t(T) \left(K \, \Phi(-d_-) - X_t \, \Phi(-d_+) \right)$$

$$d_\pm = \frac{\ln(X_t / K) \pm \frac{1}{2} \bar{\sigma}^2 (T-t)}{\bar{\sigma} \sqrt{T-t}}$$

$$\Rightarrow \boxed{V_t = K \, P_t(T) \, \Phi(-d_-) - P_t(U) \, \Phi(-d_+)}$$

$$d_\pm = \frac{\ln\left(P_t(U) / K P_t(T) \right) \pm \frac{1}{2} \bar{\sigma}^2 (T-t)}{\bar{\sigma} \sqrt{T-t}}$$

$$\frac{V_t}{P_t(u)} = \mathbb{E}^{Q^u}\left[\frac{(K - P_T(u))_+}{P_T(u)} \,\Big|\, \mathcal{F}_t \right]$$

$$= \mathbb{E}^{Q^u}\left[\left(\frac{K\, P_T(T)}{P_T(u)} - 1 \right)_+ \,\Big|\, \mathcal{F}_t \right]$$

$\hookrightarrow X_T \;,\; X_t = \dfrac{P_t(T)}{P_t(u)}$ is a Q^u

$$\frac{dX_t}{X_t} = (\;)\,dt - \sigma B_t(T)\,dW_t + \sigma B_t(u)\,dW_t$$

$$= \sigma\,(B_t(u) - B_t(T))\,dW_t^u$$

$$X_T \overset{d}{=} X_t\, e^{-\frac{1}{2}\bar{\sigma}^2 (T-t) + \bar{\sigma}\sqrt{T-t}\, Z}$$

$$Z \underset{Q^u}{\sim} \mathcal{N}(0,1)$$

$$\bar{\sigma}^2 = \frac{\sigma^2}{T-t}\int_t^T (B_s(u) - B_s(T))^2\, ds$$

$$\Rightarrow \quad V_t = K P_t(u)\, \mathbb{E}^{Q^u}\left[\left(X_T - \frac{1}{K} \right)_+ \right]$$

$$= K\, P_t(u) \left(X_t\, \Phi(d_+) - \frac{1}{K}\, \Phi(d_-) \right)$$

$\hookrightarrow 1 - \Phi(-d_+)$ $\qquad\qquad \hookrightarrow 1 - \Phi(-d_-)$

$$d_\pm = \frac{\ln(X_t K) \pm \frac{1}{2}\bar{\sigma}^2 (T-t)}{\bar{\sigma}\sqrt{T-t}}$$

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stochastic interest + equity:

$$\frac{dS_t}{S_t} = r_t\,dt + \sigma\,dW_t$$

$d[W, B]_t = \rho\,dt$

$$dr_t = \kappa(\theta - r_t)\,dt + \eta\,dB_t$$

$$\Rightarrow \frac{dP_t(T)}{P_t(T)} = r_t\,dt - \eta\,\mathcal{B}_t(T)\,dB_t$$

(W_t, B_t risk neutral B. motions)

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}(S_T - K)_+ \mid \mathcal{F}_t\right]$$

$$\frac{V_t}{P_t(T)} = \mathbb{E}_t^{\mathbb{Q}^T}\left[\frac{(S_T - K)_+}{\underbrace{P_T(T)}_{=1}}\right]$$

$$= \mathbb{E}_t^{\mathbb{Q}^T}\left[\left(\frac{S_T}{P_T(T)} - K\right)_+\right]$$

$\frac{S_T}{P_T(T)} = X_T$, $X_t = \frac{S_t}{P_t(T)}$ is a $\mathbb{Q}^T$-mtg.

$$\frac{dX_t}{X_t} = (\quad)\,dt + \sigma\,dW_t + \eta\,\mathcal{B}_t(T)\,dB_t$$

$$= \sigma\,dW_t^T + \eta\,\mathcal{B}_t(T)\,dB_t^T$$

[recall if $\frac{dY_t}{Y_t} = \sigma_t \cdot dW_t$,

then $Y_t = Y_0 \exp\left\{-\frac{1}{2}\int_0^t \|\sigma_s\|^2 ds + \int_0^t \sigma_s \cdot dW_s\right\}$

$\|\sigma_s\|^2 = \sum_{i,j=1}^n \sigma_s^i\,\sigma_s^j\,\rho_{ij}$

$d[W^i, W^j] = \rho_{ij}\,dt$]

$$X_T \stackrel{d}{=} X_t \exp\left\{-\frac{1}{2}\bar{\sigma}^2(T-t) + \bar{\sigma}\sqrt{(T-t)}\,Z\right\}$$

$$Z \underset{\mathbb{Q}^T}{\sim} \mathcal{N}(0,1)$$

$$\bar{\sigma}^2 = \frac{1}{T-t}\int_t^T \left(\sigma^2 + \eta^2 B_s^2(T) + 2\rho\sigma\eta B_s(T)\right) ds$$

so, $V_t = P_t(T)\, \mathbb{E}_t^{Q^T}\left[(X_T - K)_+ \right]$

$$= P_t(T) \left(X_T\, \Phi(d_+) - K\, \Phi(d_-)\right)$$

$$d_\pm = \frac{\ln(X_t/K) \pm \frac{1}{2}\bar{\sigma}^2 (T-t)}{\bar{\sigma}\sqrt{T-t}}$$

$$V_t = S_t\, \Phi(d_+) - K\, P_t(T)\, \Phi(d_-)$$

$$d_\pm = \frac{\ln(S_t / K\, P_t(T)) \pm \frac{1}{2}\bar{\sigma}^2 (T-t)}{\bar{\sigma}\sqrt{T-t}}$$

$$P_t = e^{-r\tau}\mathbb{E}_t^{Q}(U_T - a V_T)_+$$

$$\left.\begin{aligned} \frac{dU_t}{U_t} &= r\,dt + \sigma\,dW_t \\ \frac{dV_t}{V_t} &= r\,dt + \eta\,dB_t \end{aligned}\right) \rho$$

$$\frac{P_t}{V_t} = \mathbb{E}_t^{Q^V}\left[\left(\underbrace{\frac{U_T}{V_T}}_{=\, X_T} - a\right)_+\right]$$

X_T, $\quad X_t \triangleq \frac{U_t}{V_t}$ is a Q^V-mtg.

$$\frac{dX_t}{X_t} = (\quad)\,dt + \sigma\,dW_t - \eta\,dB_t$$

$$= \sigma\,dW_t^V - \eta\,dB_t^V$$

$$X_T \overset{d}{=} X_t \exp\left\{-\frac{1}{2}\bar{\sigma}^2(T-t) + \bar{\sigma}\sqrt{T-t}\,Z\right\}$$

$$Z \underset{Q^V}{\sim} \mathcal{N}(0,1)$$

$$\bar{\sigma}^2 = \frac{1}{T-t}\int_t^T (\sigma^2 + \eta^2 - 2\rho\sigma\eta)\,du$$

$$= (\sigma^2 + \eta^2 - 2\rho\sigma\eta)$$

$$P_t = V_t\,\mathbb{E}_t^{Q^V}\left[(X_T - a)_+\right]$$

$$= V_t \left(X_t \, \Phi(d_+) - a \, \Phi(d_-) \right)$$

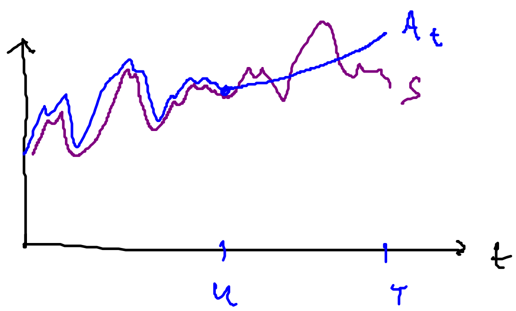
$$d_\pm = \frac{\ln(X_t / a) \pm \frac{1}{2} \tilde{\sigma}^2 (T - t)}{\tilde{\sigma} \sqrt{T - t}}$$

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dW_t$$

$$(S_T - a S_u)_+ \quad @T \qquad u < T$$

using the numeraire asset:

$$A_t = \begin{cases} S_t, & t \le u \\ S_u e^{r(t-u)}, & t > u \end{cases}$$



$$\frac{V_0}{A_0} = \mathbb{E}_0^{\mathbb{Q}^A}\left[\frac{(S_T - a S_u)_+}{A_T}\right]$$

$$= \mathbb{E}_0^{\mathbb{Q}^A}\left[\left(\frac{S_T}{A_T} - a\frac{S_u}{A_T}\right)_+\right]$$

$\hookrightarrow S_u e^{r(T-u)}$

$$= \mathbb{E}_0^{\mathbb{Q}^A}\left[\left(\frac{S_T}{A_T} - a\, e^{-r(T-u)}\right)_+\right]$$

$\hookrightarrow X_T, \quad X_t = \frac{S_t}{A_t}$ is a $\mathbb{Q}^A$-mtg.

$$\frac{dA_t}{A_t} = \begin{cases} r\,dt + \sigma\,dW_t, & t \le u \\ r\,dt. & t > u \end{cases}$$

$$= r\,dt + \underbrace{\sigma\,\mathbb{1}_{t\le u}}_{\tilde{\sigma}_t}\,dW_t$$

$$\frac{dX_t}{X_t} = (\quad)\,dt + \sigma\,dW_t - \sigma\,\mathbb{1}_{t\le u}\,dW_t$$

$$= \sigma\,(1 - \mathbb{1}_{t\le u})\,dW_t^A$$

$$= \sigma\,\mathbb{1}_{t>u}\,dW_t^A$$

$$X_T \overset{d}{=} X_0 \exp\left\{ -\frac{1}{2}\tilde{\sigma}^2 (T-0) + \tilde{\sigma}\sqrt{(T-0)}\,Z \right\}$$

$$\tilde{\sigma}^2 = \frac{1}{T-0\ \ t} \int_{0\ \ t}^{T} (\sigma\,\mathbb{1}_{s>u})^2\,ds$$

$$= \frac{1}{T}(T-u)\,\sigma^2$$

$$V_0 = A_0^{t}\left(X_0^{t}\,\Phi(d_+) - a\,e^{-r(T-u)}\,\Phi(d_-) \right)$$

$$d_\pm = \frac{\ln\left(X_0^{t} / a\,e^{-r(T-u)}\right) \pm \frac{1}{2}\tilde{\sigma}^2 (T)^{T-t}}{\tilde{\sigma}\sqrt{T}\ \ T-t}$$

$$= S_0\left(\Phi(d_+) - a\,e^{-r(T-u)}\,\Phi(d_-)\right)$$

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$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\, dW_t \longrightarrow dx_t = -\kappa x_t\, dt + \sigma\, dW_t$$

$$r_n - r_{n-1} = \kappa(\theta - r_{n-1})\Delta t + \sigma\sqrt{\Delta t}\, x_n$$

$$x_t = r_t + g_t \qquad (\text{choose } x_0 = 0 \Rightarrow g_0 = -r_0)$$

$$dx_t = dr_t + dg_t$$

$$-\kappa(r_t + g_t)\,dt + \sigma\, dW_t$$

$$= \kappa(\theta - r_t)\,dt + \sigma\, dW_t + dg_t$$

$$\Rightarrow -\kappa g_t\, dt = \kappa\theta\, dt + dg_t$$

$$\Rightarrow -\kappa(\theta + g_t)\,dt = dg_t$$

$$\Rightarrow \frac{dg_t}{\theta + g_t} = -\kappa\, dt \Rightarrow d\ln(\theta + g_t) = -\kappa\, dt$$

$$\Rightarrow \ln\left(\frac{\theta + g_t}{\theta + g_0}\right) = -\kappa t$$

$$\Rightarrow \theta + g_t = (\theta + g_0)e^{-\kappa t}$$

$$\Rightarrow g_t = (g_0 + \theta)e^{-\kappa t} - \theta$$

$$r_t = x_t + \theta - (g_0 + \theta)e^{-\kappa t}$$

$$\Rightarrow r_t = x_t + \theta + (r_0 - \theta)e^{-\kappa t}$$

$$dx_t = -\kappa x_t\, dt + \sigma\, dW_t, \quad x_0 = 0$$

$$x_{t+\Delta t} \stackrel{d}{=} x_t e^{-\kappa\Delta t} + \frac{\sigma}{\sqrt{2\kappa}}\left(1 - e^{-2\kappa\Delta t}\right)^{1/2} Z$$

- Trinomial trees are used instead
- Branching probabilities choosing to match mean and variance

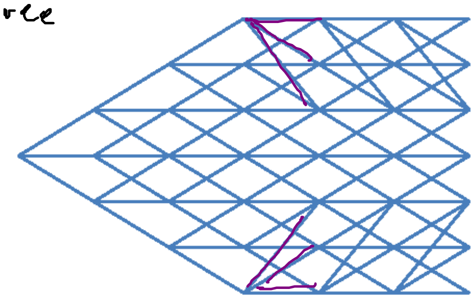
$$\mathbb{E}_t^{\mathbb{Q}}[X_{t+\Delta t} - X_t] = (e^{-\kappa\Delta t} - 1)X_t \triangleq M X_t$$

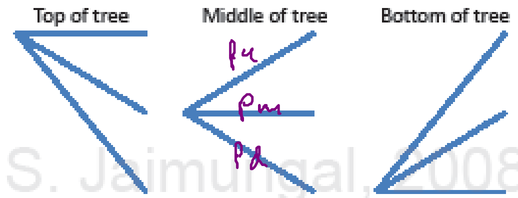
$$\mathbb{V}_t^{\mathbb{Q}}[X_{t+\Delta t} - X_t] = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa\Delta t}) \triangleq V$$

- Branch steps set to $\Delta X = \sqrt{3V}$

Zero mean-reversion level

x-tree





- Middle of tree branching probabilities:

$$
\begin{aligned}
p_u &= \tfrac{1}{6} + \tfrac{j^2M^2+jM}{2} \\
p_m &= \tfrac{2}{3} - j^2M^2 \\
p_d &= \tfrac{1}{6} + \tfrac{j^2M^2-jM}{2}
\end{aligned}
$$

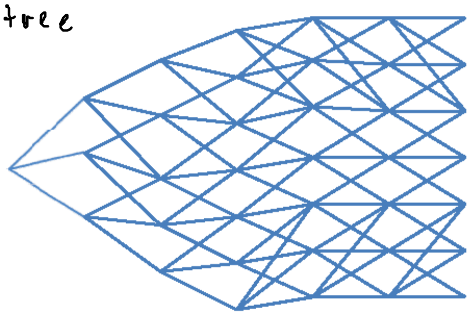
$j = 0$ is centre.

- Top and Bottom of tree branching probabilities:

$$
\begin{array}{ll}
\textit{Top} & \textit{Bottom} \\
p_u = \tfrac{7}{6} + \tfrac{j^2M^2+3jM}{2} & p_u = \tfrac{1}{6} + \tfrac{j^2M^2-jM}{2} \\
p_m = -\tfrac{1}{3} - j^2M^2 - 2jM & p_m = -\tfrac{1}{3} - j^2M^2 + 2jM \\
p_d = \tfrac{1}{6} + \tfrac{j^2M^2+jM}{2} & p_d = \tfrac{7}{6} + \tfrac{j^2M^2-3jM}{2}
\end{array}
$$

Shifted mean-reversion level

r_t-tree



$r_{T,1} \rightarrow P_{T,1} = e^{A_T(u) - r_{T,1} B_T(u)}$

$r_{T,2}$

$r_{T,3}$

$\vdots$

T