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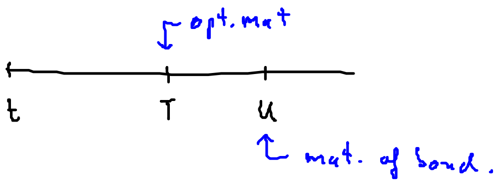
10:06 AM

Bond options: $\varphi = \underline{\underline{(P_T(u) - K)_+}}$ mat $= T$



$$V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,ds} \left(P_T(u) - K \right)_+ \,\middle|\, \mathcal{F}_t \right]$$

$$P_T(u) = e^{A_T(u) - B_T(u)\, r_T}$$

since Vasicek is affine.

$$\int_t^T r_u\,du = \hat{\theta}\left[(T-t) - \frac{1-e^{-\hat{\kappa}(T-t)}}{\hat{\kappa}} \right] + \frac{1-e^{-\hat{\kappa}(T-t)}}{\hat{\kappa}}\, r_t$$

$$+ \frac{\sigma}{\hat{\kappa}} \int_t^T \left(1 - e^{-\hat{\kappa}(T-u)} \right) d\hat{W}_u$$

$$r_T = r_t\, e^{-\kappa(T-t)} + \theta\left(1 - e^{-\kappa(T-t)}\right) + \sigma \int_t^T e^{-\kappa(T-u)}\, d\hat{W}_u$$

Clearly $\left(\int_t^T r_u\,du ,\ r_T \right)$ is a bivariate normal.

$$\mathbb{C}^{\mathbb{Q}}\left[r_T,\ \int_t^T r_u\,du \right]$$

$$= \frac{\sigma^2}{\hat{\kappa}}\, \mathbb{C}\left[\int_t^T \left(1 - e^{-\hat{\kappa}(T-u)}\right) d\hat{W}_u ,\ \int_t^T e^{-\hat{\kappa}(T-u)}\, d\hat{W}_u \right]$$

$$= \frac{\sigma^2}{\hat{\kappa}}\, \mathbb{E}\left[\int_t^T \left(1 - e^{-\hat{\kappa}(T-u)}\right) d\hat{W}_u \int_t^T e^{-\hat{\kappa}(T-u)}\, d\hat{W}_u \right]$$

$$= \frac{\sigma^2}{\hat{\kappa}}\, \mathbb{E}\left[\int_t^T \left(1 - e^{-\hat{\kappa}(T-u)}\right) e^{-\hat{\kappa}(T-u)}\, du \right]$$

$$= \frac{\sigma^2}{\hat{k}} \left(\frac{1 - e^{-\hat{k}(T-t)}}{\hat{k}} - \frac{1 - e^{-2\hat{k}(T-t)}}{2\hat{k}} \right)$$

recall ... $\exists$ a $\mathbb{Q}^A$ s.t.

$$\frac{V_t}{A_t} = \mathbb{E}^{\mathbb{Q}^A}\left[\frac{V_T}{A_T} \,\middle|\, \mathcal{F}_t \right] \iff \text{no arb.}$$

(A_t is a numeraire asset : $\times$ traded asset
$\times$ > 0 a.s)

$$A_0 \begin{cases} A_u & q \\ A_d & \end{cases}$$

$$q^A = \left(\frac{A_u/A_0}{M_u/M_0}\right) q \quad \checkmark$$

$$1 - q^A = \left(\frac{A_d/A_0}{M_d/M_0}\right)(1-q)$$

no arb $\iff$

$\exists$ $\mathbb{Q}$ s.t. $\dfrac{V_t}{M_t} = \mathbb{E}^{\mathbb{Q}}\left[\dfrac{V_T}{M_T} \,\middle|\, \mathcal{F}_t \right]$

$$\frac{V_0}{M_0} = \frac{V_u}{M_u} q + \frac{V_d}{M_d}(1-q)$$

$$\frac{V_0}{A_0} = \frac{1/A_0}{M_u/M_0} V_u \, q + \frac{1/A_0}{M_d/M_0} V_d (1-q)$$

$$= \underbrace{\left(\frac{A_u/A_0}{M_u/M_0} q\right)}_{q^A} \frac{V_u}{A_u} + \underbrace{\left(\frac{A_d/A_0}{M_d/M_0}(1-q)\right)}_{1-q^A} \frac{V_d}{A_d}$$

note: $\dfrac{A_u/A_0}{M_u/M_0} q + \dfrac{A_d/A_0}{M_d/M_0}(1-q)$

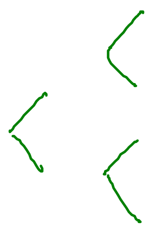
$$= \frac{M_0}{A_0}\left(\frac{A_u}{M_u} q + \frac{A_d}{M_d}(1-q) \right)$$

$$= \frac{M_0}{A_0} \cdot \frac{A_0}{M_0} = 1$$

so how to choose R-N ?

$$\left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right) = \frac{A_1/A_0}{M_1/M_0}$$

$$\left[\quad \left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right)(\omega) \quad d\mathbb{Q}(\omega) = d\mathbb{Q}^A(\omega) \quad \right]$$



$$\left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right) = \left(\frac{A_2/A_1}{M_2/M_1}\right)\left(\frac{A_1/A_0}{M_1/M_0}\right) = \frac{A_2/A_0}{M_2/M_0}$$

guess in continuous time:

$$\left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right)_T = \frac{A_T/A_0}{M_T/M_0}$$

need to show that

① $\left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right)_T$ is a bonified measure change.

- $\mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right] = 1 \rightarrow \mathbb{E}^{\mathbb{Q}}\left[\frac{A_T/A_0}{M_T/M_0}\right] = \frac{M_0}{A_0} \cdot \mathbb{E}^{\mathbb{Q}}\left[\frac{A_T}{M_T}\right]$

$$= \frac{M_0}{A_0} \cdot \frac{A_0}{M_0} = 1$$

- > 0 a.s. $\rightarrow$ trivial b.c.
  - A is a numeraire asset
  - $M > 0$ a.s.

② no arb $\Leftrightarrow \frac{V_t}{A_t} = \mathbb{E}^{\mathbb{Q}^A}\left[\frac{V_T}{A_T}\right]$.

need to show that $\frac{V_t}{A_t}$ is a $\mathbb{Q}^A$-mtg.

so compute: $s < t < T$

$$\mathbb{E}^{\mathbb{Q}^A}\left[\frac{V_t}{A_t} \,\Big|\, \mathcal{F}_s\right] = \frac{\mathbb{E}^{\mathbb{Q}}\left[\frac{V_t}{A_t}\cdot\left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right)_T \,\Big|\, \mathcal{F}_s\right]}{\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right)_T \,\Big|\, \mathcal{F}_s\right]}$$

$$\text{numer} = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\frac{V_t}{A_t}\cdot\left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right)_T \,\Big|\, \mathcal{F}_t\right] \,\Big|\, \mathcal{F}_s\right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\frac{V_t}{A_t}\cdot\left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right)_t \,\Big|\, \mathcal{F}_s\right]$$

$$\left(\begin{aligned} &\eta_t = \left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right)_t = \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right)_T \,\Big|\, \mathcal{F}_t\right] \quad \text{is a Doob-mtg} \\ &\mathbb{E}^{\mathbb{Q}}[\eta_t \mid \mathcal{F}_s] = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right)_T \,\Big|\, \mathcal{F}_t\right] \,\Big|\, \mathcal{F}_s\right] \qquad s < t \\ &\qquad = \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right)_T \,\Big|\, \mathcal{F}_s\right] = \eta_s. \end{aligned}\right)$$

$$\Rightarrow \mathbb{E}^{\mathbb{Q}^A}\left[\frac{V_t}{A_t} \,\Big|\, \mathcal{F}_s\right] = \frac{\mathbb{E}^{\mathbb{Q}}\left[\frac{V_t}{A_t}\cdot\overbrace{\left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right)_t}^{\frac{A_t/A_0}{M_t/M_0}} \,\Big|\, \mathcal{F}_s\right]}{\underbrace{\left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right)_s}_{\frac{A_s/A_0}{M_s/M_0}}}$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\frac{V_t}{M_t} \,\Big|\, \mathcal{F}_s\right]\cdot\frac{M_s}{A_s}$$

$$= \frac{V_s}{M_s}\cdot\frac{M_s}{A_s} = \frac{V_s}{A_s}$$

so no arb $\iff$ $\dfrac{V_t}{A_t}$ is a $\mathbb{Q}^A$-mtg.

$\frac{V_t}{A_t}$ is a $\mathbb{Q}^A$-mtg

$$\begin{tikzcd} \mathbb{P} \arrow[r, leftrightarrow, "\lambda"] & \text{PDE} \arrow[r, leftrightarrow] & \mathbb{Q} \arrow[r, leftrightarrow] \arrow[d] & \mathbb{Q}^A \arrow[u] \\ & & \frac{V_t}{M_t} \text{ is a } \mathbb{Q}\text{-mtg} & \end{tikzcd}$$

dynamic hedging no-arb

Feynman Kac

$$\mathbb{P} \xrightarrow{\lambda} \qquad \mathbb{Q} \longrightarrow \mathbb{Q}^A$$

$$\left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right)_T = \frac{A_T / A_0}{M_T / M_0}$$

$$\eta_t = \left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right)_t$$

recall Girsanov's Thm says that if W_t is a $\mathbb{Q}$-B.mtn then

$$\hat{W}_t = \int_0^t \gamma(s, W_s)\, ds + W_t$$

is a $\mathbb{Q}^*$-B mtn where $\mathbb{Q}^*$ is defined via the R-N:

$$\zeta_T = \left(\frac{d\mathbb{Q}^*}{d\mathbb{Q}}\right)_T = \exp\left\{ -\frac{1}{2}\int_0^T \gamma^2(s, W_s)\, ds - \int_0^T \gamma(s, W_s)\, dW_s \right\}$$

Doleans - Dades exponential

$$(*) \quad \frac{d\zeta_t}{\zeta_t} = -\gamma(t, W_t)\, dW_t$$

Ito's lemma: if $dX_t = \mu_t^X\, dt + \sigma_t^X\, dW_t$

then $$dF(t, X_t) = \left(\partial_t f + \mu_t^X \partial_x f + \tfrac{1}{2}(\sigma_t^X)^2 \partial_{xx} f \right) dt + \sigma_t^X \partial_x f\, dW_t$$

check (*)

$$d\ln \zeta_t = \left(0 + 0\left(\frac{1}{\zeta_t}\right) + \frac{1}{2}\zeta_t^2\gamma^2(t,W_t)\left(-\frac{1}{\zeta_t^2}\right)\right)dt$$

$$- \zeta_t\gamma(t,W_t)\cdot\frac{1}{\zeta_t}\,dW_t$$

$$= -\frac{1}{2}\gamma^2(t,W_t)\,dt - \gamma(t,W_t)\,dW_t$$

$$\ln\zeta_t - \ln\zeta_0 = -\frac{1}{2}\int_0^t \gamma^2(s,W_s)\,ds - \int_0^t \gamma(s,W_s)\,dW_s$$

$$\Rightarrow \quad \zeta_t = \exp\left\{-\frac{1}{2}\int_0^t \gamma^2(s,W_s)\,ds - \int_0^t \gamma(s,W_s)\,dW_s\right\}$$

Let's compute $d\eta_t$ ...

$$d\eta_t = d\left(\frac{A_t/A_0}{M_t/M_0}\right) = \frac{M_0}{A_0}\,d\left(\frac{A_t}{M_t}\right)$$

recall that $\dfrac{dA_t}{A_t} = r_t\,dt + \sigma_t^A\,dW_t$

$$\rightarrow \quad A_t = A_0\exp\left\{\int_0^t\left(r_s - \tfrac{1}{2}(\sigma_s^A)^2\right)ds + \int_0^t \sigma_s^A\,dW_s\right\}$$

$$\frac{dM_t}{M_t} = r_t\,dt$$

$$d\left(\frac{A_t}{M_t}\right) = \sigma_t^A\left(\frac{A_t}{M_t}\right)dW_t$$

$$d\left(\frac{A_t}{M_t}\right) = d\left(e^{-\int_0^t r_s\,ds}A_t\right)$$

$$= d\left(e^{-\int_0^t r_s\,ds}\right)A_t + e^{-\int_0^t r_s\,ds}\,dA_t$$

$$+ \underbrace{\left[e^{-\int_0^t r_s\,ds}, A_t\right]}_{\to 0}$$

$$= -r_t\frac{A_t}{M_t}\,dt + \frac{1}{M_t}\left(r_t A_t\,dt + \sigma_t^A A_t\,dW_t\right)$$

$$= \sigma_t^A\left(\frac{A_t}{M_t}\right)dW_t$$

$$\Rightarrow \quad d\eta_t = \frac{M_0}{A_0} \cdot \sigma_t^A \cdot \frac{A_t}{M_t} \, dW_t = \eta_t \, \sigma_t^A \, dW_t$$

$$\Rightarrow \quad \boxed{\frac{d\eta_t}{\eta_t} = \sigma_t^A \, dW_t}$$

typically

$\sigma^A(t, A_t)$

$$\therefore \text{ Girsanov's } \Rightarrow \boxed{d\hat{W}_t = -\sigma_t^A \, dt + dW_t}$$

is a $\mathbb{Q}^A$ - B. mtn.

bond option:

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,ds} (P_T(u) - K)_+ \mid \mathcal{F}_t \right]$$

introduce a new $\overset{\text{numeraire}}{\text{measure}}$ $\mathbb{Q}^A$

$$\frac{V_t}{A_t} = \mathbb{E}^{\mathbb{Q}^A}\left[\frac{(P_T(u) - K)_+}{A_T} \mid \mathcal{F}_t \right]$$

two natural choices: $P_t(T)$ or $P_t(u)$

s.c. $P_T(T) = 1$

$$\text{so} \quad V_t = P_t(T)\, \mathbb{E}^{\mathbb{Q}^A}\left[\frac{(P_T(u) - K)_+}{\underbrace{P_T(T)}_{\to 1}} \mid \mathcal{F}_t \right]$$

need $\mathbb{E}^{\mathbb{Q}^A}\left[(P_T(u) - K)_+ \mid \mathcal{F}_t \right]$

we know that $P_t(u) = e^{A_t(u) - B_t(u) r_t}$

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\, dW_t$$

$\llcorner$ $\mathbb{Q}$ - B.mtn.

$$dP_t(u) = \underbrace{(\partial_t + \mathcal{L}) P}_{r_t P_t(T)}\, dt + \sigma \cdot (-B_t(u)) P_t(T)\, dW_t$$

$$\Rightarrow \frac{dP_t(u)}{P_t(u)} = r_t\, dt - \sigma B_t(u)\, dW_t$$

$$\frac{dP_t(T)}{} = r_t\, dt - \sigma B_t(T)\, dW_t$$

$P_t(T)$ numeraire vol

$$\Rightarrow \quad d\hat{W}_t = \sigma B_t(T)\,dt + dW_t$$

$\mathbb{Q}^T$ - B. mtn.

Forward-neutral measure

$$\Rightarrow \quad \frac{dP_t(u)}{P_t(u)} = r_t\,dt - \sigma B_t(u)\left(d\hat{W}_t - \sigma B_t(T)\,dt\right)$$

$$= \left(r_t + \sigma^2 B_t(u) B_t(T)\right) dt$$

$$- \sigma B_t(u)\,d\hat{W}_t$$

$$dr_t = \left(\kappa(\theta - r_t) - \sigma^2 B_t(T)\right) dt + \sigma\,d\hat{W}_t$$

recall want $\mathbb{E}^{\mathbb{Q}}\left[\left(P_T(u) - K\right)_+ \mid \mathcal{F}_t\right]$

$$\frac{P_T(u)}{P_T(T)} = X_T$$

$P_T(T) = 1$

$X_t = \dfrac{P_t(u)}{P_t(T)}$ is a $\mathbb{Q}^T$ - mtg

$$\therefore \quad \frac{dX_t}{X_t} = \Big(\underbrace{-\sigma B_t(u)}_{\text{vol of } P_t(u)} + \underbrace{\sigma B_t(T)}_{-\text{ vol of } P_t(T)}\Big)\,d\hat{W}_t$$

From affine form

$$X_t = \exp\left\{\left(A_t(u) - A_t(T)\right) - \left(B_t(u) - B_t(T)\right) r_t\right\}$$

$$dX_t = (0)\,dt + \sigma\left(-B_t(u) + B_t(T)\right) X_t\,d\hat{W}_t$$

deterministic

$$\text{so} \quad \frac{dX_t}{X_t} = \sigma h(t)\,d\hat{W}_t$$

X_t

$$\Rightarrow \quad X_T = X_t \, e^{-\frac{1}{2}\sigma^2 \underbrace{\int_t^T h^2(s)\,ds}_{\bar{\sigma}^2 (T-t)} + \int_t^T \sigma h(s)\, d\hat{W}_s}$$

$$\overset{d}{=} X_t \exp\left\{ -\frac{1}{2} \bar{\sigma}^2 (T-t) + \bar{\sigma}\sqrt{T-t}\; Z \right\}$$

$$Z \underset{\mathbb{Q}^T}{\sim} \mathcal{N}(0,1)$$

$$\mathbb{E}^{\mathbb{Q}^T}\left[(X_T - K)_+ \mid \mathcal{F}_t \right]$$

$$= \underline{X_t\, \Phi(d_+) - K\, \Phi(d_-)}$$

$$d_\pm = \frac{\ln(X_t/K) \pm \frac{1}{2}\bar{\sigma}^2 (T-t)}{\bar{\sigma}\sqrt{T-t}}$$

$$V_t = P_t(T) \times (\qquad\qquad)$$