IMPA Commodities Course : Numerical Methods

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Basic MC Methods

- For option valuation, an expectation is the basic quantity

$$V_t = e^{-r(T-t)}\,\mathbb{E}_t^{\mathbb{Q}}\left[\varphi(F_T(U))\right]$$

- For value-at-risk calculations, quantiles are the object of interest

$$VaR_p \triangleq \max\left\{X : \mathbb{P}\left(\sum_i a_i V_i > X\right) > p\right\} - \mathbb{E}^{\mathbb{P}}\left[\sum_i a_i V_i\right]$$

- For simulation, the price path $\{F_{t_1}(U), F_{t_2}(U), \dots\}$ is required
- In all cases Monte Carlo simulation can assist

Basic MC Methods

- Idea: simulate from the distribution of relevant quantity and compute average
- Example: Price a 1-year floating strike Asian option on the 1.25-year forward contract with averaging occurring over the last month of the contract using the Schwartz model
  - The payoff of this option is

$$\varphi = \left(F_1(1.25) - \sum_{i=1}^{20} F_{1+(i-20)/251}(1.25) \right)_+$$

  - Need to simulate the forward price for all days in final month
  - Simulate in one step the forward price at time $1 - 20/251$
  - Simulate daily time-steps ($\Delta t = 1/251$) within month
  - Compute pay-off and discount
  - Average over many paths

Brownian Bridges

- Rather than simulating entire path, can simulate the path in a **progressive refinement** manner
- A **Brownian bridge** is a way of generating a Brownian path conditional on the end points
  - Given $X(0) = x_0$ and $X(t) = x_t$ generate $X(s)$ for $0 < s < t$.
- The joint distribution of $X(t), X(s)$ given $X(0)$ is a bivariate normal

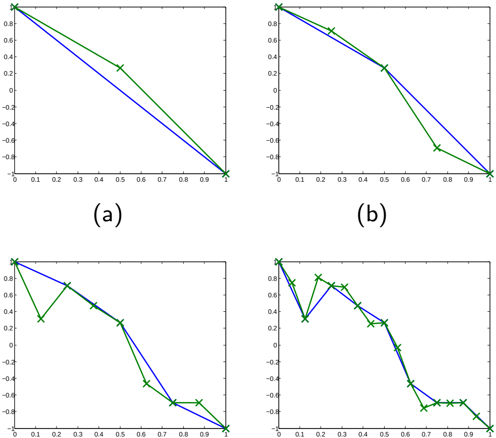
$$\left(\begin{array}{c} X(s) \\ X(t) \end{array}\right)\Bigg|_{X_0=x_0} \sim \mathcal{N}\left(\left(\begin{array}{c} x_0 \\ x_0 \end{array}\right); \left(\begin{array}{cc} s & s \\ s & t \end{array}\right)\right)$$

- Can then show that,

$$X(s)|_{X(0)=x_0, X(t)=x_t} \sim \mathcal{N}\left(x_0 + \frac{s}{t}(x_t - x_0); \sigma^2 \frac{(t-s)s}{t}\right)$$

Brownian Bridges

A Bridge refinement example: 4 refinement steps

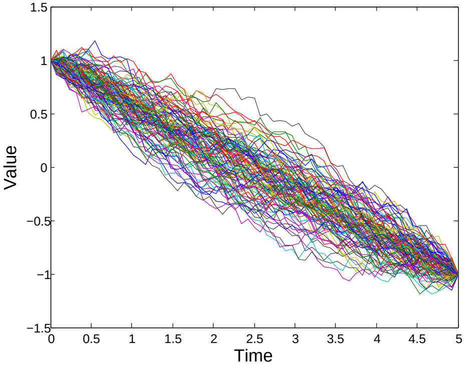


(a) (b)

(c) (d)

Brownian Bridges

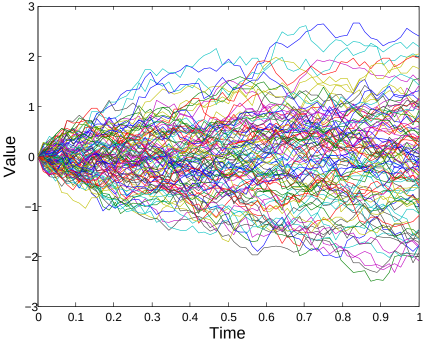
Many Brownian Bridge paths: 6 refinement steps, 100 paths, $X_0 = 1$, $X_1 = -1$, $\sigma = 0.2$



Brownian Bridges

To generate Brownian sample paths using a Brownian Bridge:

1. generate random sample of $X(t)$ given $X(0)$:
   $X(t)|_{X(0)=x_0} \sim \mathcal{N}(x_0; t)$
2. build bridge from $X(0) = x_0$ to $X(t) = x_t$
3. repeat from step 1



Mean-Reverting Bridges

- For mean-reverting process

$$dX_t = \kappa(\theta - X_t)\,dt + \sigma\,dW_t$$

- The joint of X_{t_1}, X_s given X_{t_0} is

$$\left(\begin{array}{c} X_s \\ X_{t_1} \end{array}\right)\Bigg|_{X_{t_0}} \sim \mathcal{N}\left(\left(\begin{array}{c} \theta + e^{-\kappa\,(s-t_0)}(X_{t_0} - \theta) \\ \theta + e^{-\kappa\,(t-t_0)}(X_{t_0} - \theta) \end{array}\right) ; \Sigma\right)$$

where

$$\Sigma = \frac{\sigma^2}{2\kappa}\left(\begin{array}{cc} 1 - e^{-2\kappa\,(s-t_0)} & 1 - e^{-2\kappa\,(s-t_0)} \\ 1 - e^{-2\kappa\,(s-t_0)} & 1 - e^{-2\kappa\,(t-t_0)} \end{array}\right)$$

Mean-Reverting Bridges

- Can then show that

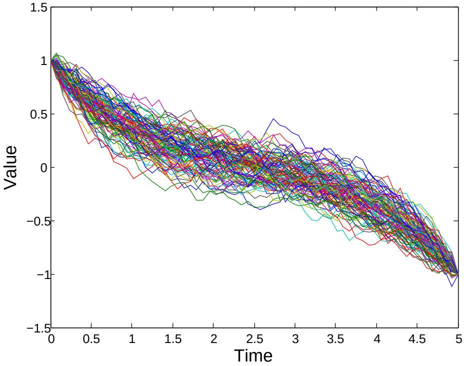
$$X_s\big|_{X_{t_0},X_{t_1}} \sim \mathcal{N}\left(m; v\right)$$

with

$$\begin{aligned} m &= e^{-\kappa(s-t_0)}\Big[X_0 + \theta(e^{\kappa\,(s-t_0)} - 1) \\ &\quad + \frac{e^{2\kappa\,(s-t_0)} - 1}{e^{2\kappa\,(t-t_0)} - 1}(e^{-\kappa(t_1-t_0)}\,X_{t_1} - (X_{t_0} + \theta(e^{\kappa\,(t_1-t_0)} - 1)))\Big] \\ v &= \frac{\sigma^2}{2\kappa}\,(e^{2\kappa\,(t-s)} - 1)\,\frac{e^{2\kappa\,(s-t_0)} - 1}{e^{2\kappa\,(t-t_0)} - 1} \end{aligned}$$

Mean-Reverting Bridges

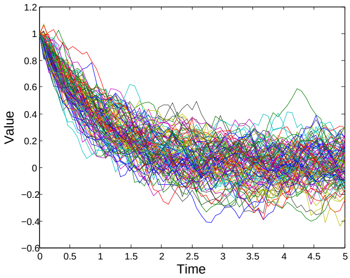
Many Mean-Reverting Bridge paths: 6 refinement steps, 100 paths, $X_0 = 1$, $X_1 = -1$, $\theta = 0$, $\kappa = 1$, $\sigma = 0.2$



Mean-Reverting Bridges

To generate Mean-Reverting sample paths using a Mean-Reverting Bridge:

1. generate random sample of $X(t)$ given $X(0)$
2. build bridge from $X(0) = x_0$ to $X(t) = x_t$
3. repeat from step 1



6 refinement steps, 100 paths, $X_0 = 1$, $\theta = 0$, $\kappa = 1$, $\sigma = 0.2$

Least Squares Monte Carlo

- **Carrier**(1994) and **Longstaff & Schwartz** (2000) developed the **least-squares Monte Carlo** method for valuing early exercise clauses.
- Basic idea
  1. Generate sample paths forward in time
  2. Place payoff at end nodes
  3. Compute discounted value of option
  4. Estimate conditional expectation by projection onto basis functions
  5. Determine optimal exercise point using basis functions
  6. Repeat from step 3

Least Squares Monte Carlo

Example: American Put strike= 1, spot= 1, $r = 0.05$:

Asset prices

Path	t=0	t=1	t=2	t=3
1	1	0.95	0.94	0.82
2	1	0.97	1.21	1.15
3	1	0.96	0.91	0.87
4	1	0.84	1.20	0.87
5	1	0.93	0.90	0.91
6	1	1.03	0.99	1.01

Least Squares Monte Carlo

Example: American Put strike= 1, spot= 1, $r = 0.05$:

t=2 asset prices	t=3 payoff
0.94	**0.18**
1.21	0
0.91	**0.13**
1.20	0.13
0.90	**0.09**
0.99	**0**

- Compute payoff at $t = 3$
- Focus only on paths which are in the money at $t = 2$

Least Square Monte Carlo

- Compute discounted value of payoffs at time $t = 2$

t = 2 discounted payoff	t=2 asset prices	t=3 payoff
0.17	**0.94**	**0.18**
0	1.21	0
0.12	**0.91**	**0.13**
0.12	1.20	0.13
0.08	**0.90**	**0.09**
0	**0.99**	**0**

- Regress discounted payoff onto asset prices at $t = 2$ using basis functions (e.g. 1, S, S^2):

$$\overline{V}_2(S) = -55.04 + 117.7\,S - 62.75\,S^2$$

- Regression gives estimate of $\mathbb{E}[e^{-r\Delta t} V_{t+dt} | S_t]$

Least Square Monte Carlo

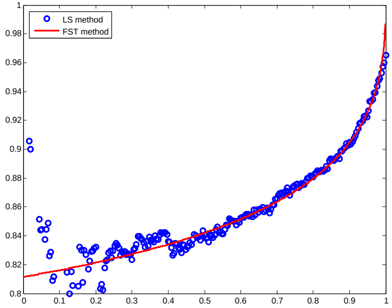
- Compare estimate discounted expectation with immediate exercise value

t = 2 est. disc. exp.	t=2 asset prices	t=2 exercise value	t=2 est. option price
0.1697	**0.94**	**0.06**	**0.17**
-	1.21	0	0
0.1208	**0.91**	**0.09**	**0.12**
-	1.20	0	0.12
0.0795	**0.90**	**0.10**	**0.10 x**
0.0001	**0.99**	**0.01**	**0.01 x**

- In this example last two branches are optimal to exercise
- Notice that the realized value at node $t = 2$ are used when going backwards, not the estimate of the conditional expectation

Least Square Monte Carlo

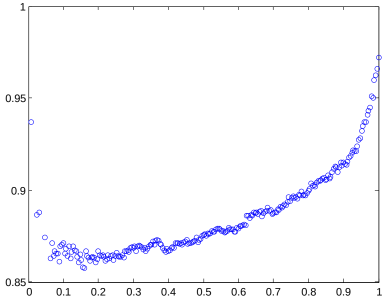
- Continue working backwards to obtain estimated prices at $t = 1$ and then $t = 0$
- Example: American put strike = 1, term = 1, r = 5%, $\sigma = 20\%$



- 250 steps, 300,000 sample paths

Least Square Monte Carlo

- Example: American put strike = 1, term = 1, r = 5%, $\kappa = 1$, $\theta = 0$, $\sigma = 20\%$



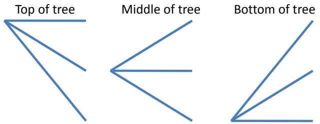
- 250 steps, 300,000 sample paths

Tress

- Binomial trees are not appropriate for commodities due to mean-reversion
- Trinomial trees are used instead
- Branching probabilities choosing to match mean and variance

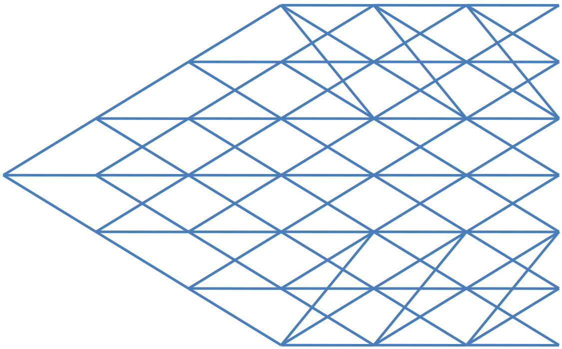
$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}[X_{t+\Delta t} - X_t] &= (e^{-\kappa \Delta t} - 1) X_t \triangleq M\, X_t \\ \mathbb{V}_t^{\mathbb{Q}}[X_{t+\Delta t} - X_t] &= \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa\Delta t}) \triangleq V\end{aligned}$$

- Branch steps set to $\Delta X = \sqrt{3V}$
- Tree is cut at high and low values to avoid negative probabilities



Tress

Zero mean-reversion level



Tress

- Middle of tree branching probabilities:

$$\begin{aligned} p_u &= \tfrac{1}{6} + \tfrac{j^2M^2 + jM}{2} \\ p_m &= \tfrac{2}{3} - j^2M^2 \\ p_d &= \tfrac{1}{6} + \tfrac{j^2M^2 - jM}{2} \end{aligned}$$

- Top and Bottom of tree branching probabilities:

$$Top$$

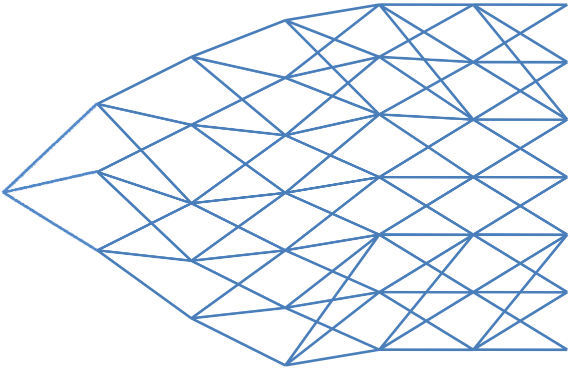
$$\begin{aligned} p_u &= \tfrac{7}{6} + \tfrac{j^2M^2 + 3jM}{2} \\ p_m &= -\tfrac{1}{3} - j^2M^2 - 2jM \\ p_d &= \tfrac{1}{6} + \tfrac{j^2M^2 + jM}{2} \end{aligned}$$

$$Bottom$$

$$\begin{aligned} p_u &= \tfrac{1}{6} + \tfrac{j^2M^2 - jM}{2} \\ p_m &= -\tfrac{1}{3} - j^2M^2 + 2jM \\ p_d &= \tfrac{7}{6} + \tfrac{j^2M^2 - 3jM}{2} \end{aligned}$$

Tress

Shifted mean-reversion level



For simple mean-reversion will shift via $\theta + (\ln S_0 - \theta) e^{-\kappa\, t} + X_t$

Tress

Comparison of LSM and Trinomial model