Tail area approximations: no nuisance parameters

- Assume $(\hat{\theta}, a)$ is a one to one function of the minimal suficient statistic, and a is ancillary, for an i.i.d. sample of size n with log-likelihood function $\ell(\theta; y) = \ell(\theta; \hat{\theta}, a) + C$. Then

$$f(\hat{\theta}; \theta | a) = c\frac{1}{\sqrt{2\pi}} |nj(\hat{\theta})|^{1/2} e^{\ell(\theta) - \ell(\hat{\theta})} \{1 + O(n^{-3/2})\}.$$

- Let

$$\begin{aligned} r &= \mathrm{sgn}(\hat{\theta} - \theta)\sqrt{2\{\ell(\hat{\theta}) - \ell(\theta)\}} \\ u &= \{nj(\hat{\theta})\}^{-1/2}\{\ell_{;\hat{\theta}}(\hat{\theta}; \hat{\theta}, a) - \ell_{;\hat{\theta}}(\theta; \hat{\theta}, a)\} \end{aligned}$$

Then

$$\begin{aligned} F(\hat{\theta}; \theta | a) = F(r; \theta | a) &= \{\Phi(r) + \phi(r)\left(\tfrac{1}{r} - \tfrac{1}{u}\right)\}\{1 + O(n^{-3/2})\} \\ &= \Phi(r^*)\{1 + O(n^{-3/2})\} \end{aligned}$$

, where $r^* = r + r^{-1}\log(u/r)$.

- The proof uses the results

  1. $c = 1 + O(n^{-1})$ (which we have proved only for exponential models)
  2. $d(\frac{1}{u} - \frac{1}{r}) = \frac{b}{n} dr$

- If the underlying density for Y_i is $\exp\{\theta y_i - c(\theta) - d(y_i)\}$, then

$$u = (\hat{\theta} - \theta)\{nj(\hat{\theta})\}^{1/2}.$$

- If the underlying density for Y_i is $f_0(y_i - \theta)$ then

$$u = \ell'(\theta)\{nj(\hat{\theta})\}^{-1/2}.$$