p^* and the saddlepoint approximation

As usual assume $Y = (Y_1, \ldots, Y_n)$, where $Y_1, \ldots, Y_n$ are independent, identically distributed random variables, from a density function $f(y)$ with moment generating function $M_Y(t) = Ee^{tY}$ and cumulant generating function $K_Y(t) = \log M_Y(t)$. Let $S_n = Y_1 + \ldots + Y_n$.

The saddlepoint approximation to the density of S_n is

$$\hat{f}_{S_n}(s) = \frac{c}{\sqrt{2\pi}} \frac{1}{\{nK_Y''(\hat{\phi})\}^{1/2}} \exp\{nK_Y(\hat{\phi}) - \hat{\phi}s\} \tag{1}$$

where $\hat{\phi}$ satisfies the equation $nK_Y'(\hat{\phi}) = s$.

- (1) has been renormalized and has relative error $O(n^{-3/2})$.
- a simple change of variables gives a saddlepoint approximation for $\bar{Y}_n = S_n/n$
- if Y_i are p-dimensional vectors, then $M_Y(t) = E\exp(t^T Y)$ and

  $$\hat{f}_{S_n}(s) = \frac{c}{\sqrt{2\pi}^p} \frac{1}{\{n|K_Y''(\hat{\phi})|\}^{1/2}} \exp\{nK_Y(\hat{\phi}) - \hat{\phi}^T s\} \tag{2}$$

  where $K_Y'(\phi)$ is a vector and $K_Y''(\phi)$ is a $p \times p$ matrix.
- If $f(y) = \exp\{\theta^T y - k(\theta) - d(y)\}$, then $K(\phi) = k(\theta + \phi) - k(\theta)$, $\hat{\phi} = \hat{\theta} - \theta$, and (1) becomes

  $$\hat{f}_{\hat{\Theta}}(\hat{\theta}) = \frac{c}{\sqrt{2\pi}^p} |nj(\hat{\theta})|^{1/2} \exp\{\ell(\theta) - \ell(\hat{\theta})\} \tag{3}$$

  using the change of variable $nk'(\hat{\theta}) = s$. In (3) $\ell(\theta) = \ell(\theta; y_1, \ldots, y_n) = \ell(\theta; \hat{\theta}) + C = \theta^T s - nk(\theta)$.