Solutions

STAC63 Final 2022

Any results established in the class or in the Exercises, appropriately referenced, can be used as part of solving these questions.

1 (a) (5 marks) Suppose you need to generate values from a distribution with density

$$f(x) = \begin{cases} cx^3 & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

where c is the normalizing constant. Determine an algorithm to do this.

Can do this by inversion or rejection for full marks. (give lots of part marks)

Inversion $\int_0^x z^3 dz = \frac{z^4}{4}\Big|_0^x = \frac{x^4}{4}$ so with $x = 2$ we have that $c = 1/4$ and the cdf is given $F(x) = \frac{x^4}{16}$ for $0 \le x \le 2$ and so for $u \in [0,1]$ we have $F^{-1}(u) = 2u^{1/4}$. Therefore, generate $U \sim U(0,1)$ and put $X = 2U^{1/4}$.

or

Rejection

Let $g(x) = \frac{1}{2}$ for $0 \le x \le 2$ which is the $U(0,2)$ density. Then $f(x) = \frac{x^3}{4} \le 4g(x)$ for all $x \in (0,2)$. So we can use the rejection algorithm by generating $Y \sim g$ by generating $U_1 \sim U(0,1)$ and putting $Y = 2U_1$ and independently generating $U_2 \sim U(0,1)$ and putting $X = Y$ whenever $U_2 g(Y) = \frac{U_2}{2} > \frac{Y^3}{4}$. If this inequality fails then we generate new values of (Y, U_2) until it is satisfied.

1 (b) (5 marks) Suppose you are asked to approximate $E(\exp(\cos(X)))$ where $X \sim f$ with f as in 1(a). Show how you would construct a Monte Carlo estimate of this quantity and also how you would assess the error in your estimate.

Generate $X_1, \ldots, X_n \sim f$ as described in 1(a). Then estimate $E(\exp(\cos X))$ by

2 $\bar{I}_n = \frac{1}{n} \sum_{i=1}^{n} \exp\{\cos(X_i)\}$. The error in the estimate is assessed by

3 quoting the interval $I_n \pm \frac{3}{\sqrt{n}} SD_n$ where

$SD_n^2 = \frac{1}{n} \sum_{i=1}^{n} \exp\{2\cos X\} - I_n^2$.

2 (a) (5 marks) Suppose that $X_1, \ldots, X_n$ is a sample (*i.i.d.*) from a $U(0,1)$ distribution and let $X_{(n)} = \max(X_1, \ldots, X_n)$. Determine the distribution function of $X_{(n)}$ and then prove that $X_{(n)} \xrightarrow{d} 1$.

$$F_{X_{(n)}}(x) = P(\max(X_1, \ldots, X_n) \le x)$$
$$= P(X_1 \le x, \ldots, X_n \le x)$$
$$= \prod_{i=1}^{n} P(X_i \le x) \quad \text{by independence}$$
$$= \begin{cases} \prod_{i=1}^{n} x^n & \text{when } 0 \le x \le 1 \\ 0 & x < 1 \\ 1 & x > 1 \end{cases}$$

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$$\text{Then } \lim_{n \to \infty} F_{X_{(n)}}(x) = \begin{cases} 0 & x < 1 \\ 1 & x \geq 1 \end{cases}$$

which implies $X_{(n)} \xrightarrow{d} 1$ since the limit is the cdf of the distribution degenerate at 1. and the limit holds at every x.

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2 (b) (5 marks) Suppose that $X_1, \ldots, X_n$ is a sample as in 2(a). What constants a and b guarantee that

$$Y_n = \frac{\sqrt{n}}{b}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - a\right) \xrightarrow{d} Z \sim N(0,1)?$$

$E(X_i) = 1/2$, $E(X_i^2) = \int_0^1 x^2\,dx = \frac{1}{3}$ so

3 $Var(X_i) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$. Therefore

2 by the CLT $a = \frac{1}{2}$ and $b = \frac{1}{\sqrt{12}}$

guarantee that this limit holds.

2 (c) (5 marks) Suppose that Y_n is as in 2(b) and $W_n = \exp(Y_n)$. Determine an asymptotic distribution for W_n that can be used to approximate probabilities for W_n.

By the delta theorem we have that

$$\frac{\sqrt{n}}{b}\left(\exp(Y_n) - \exp(a)\right) \xrightarrow{D} \left(\left(\frac{\partial \exp\{y\}}{\partial y}\right)\Big|_{y=a}\right) Z$$

3 where $Z \sim N(0,1)$ and $\frac{\partial \exp\{y\}}{\partial y}\Big|_{y=0} = e^{0} = 1$

1 Therefore $\frac{\sqrt{n}}{b}\left(\exp(Y_n) - \exp(0)\right) \approx N(0, 1)$

which implies we can approximate

$$P(W_n \le w) = P\left(\frac{\sqrt{n}}{b}\left(W_n - \exp\{0\}\right) \le \frac{\sqrt{n}}{b}\left(w - e^{0}\right)\right)$$

1 $$= P\left(e^{-0}\frac{\sqrt{n}}{b}\left(W_n - \exp\{0\}\right) \le e^{-a}\frac{\sqrt{n}}{b}\left(w - e^{0}\right)\right)$$

$$\approx \Phi\left(\frac{\sqrt{n}}{b}\left(w - 1\right)\right).$$

3 (a) (5 marks) Suppose the state space $\mathcal{S}$ for a stochastic process is finite. Is it possible to have $\lim_{n\to\infty} p_{ij}^{(n)} = 0$ for all $i, j \in \mathcal{S}$. Justify your conclusion.

No it is not possible because

2 $1 = \sum_{j \in S} p_{ij}^{(n)}$ for every i and n.

So if $\lim_{n\to\infty} p_{ij}^{(n)} = 0$ $\forall i,j$ we would

3 have $1 = \lim_{n\to\infty} \sum_{j\in S} p_{ij}^{(n)} = 0$ $\otimes$.

3 (b) (5 marks) Suppose a process $\{X_n : n \in \mathbb{N}_0\}$ has state space $\mathcal{S} = \mathbb{N}$ and $X_0 = 1, X_i = X_{i-1} + 1$ for $i \geq 1$. Is this a Markov chain and, if it is, what is the transition probability matrix P?

While $X_0 = 1, X_1 = 1, \ldots, X_{n-1} = n-1$ we have that $P(X_n = i \mid X_0 = 1, \ldots, X_{n-1} = n-1)$

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$$= \begin{cases} 1 & i = n \\ 0 & \text{otherwise} \end{cases}$$

$$= P(X_n = i \mid X_{n-1} = n-1)$$

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and so this is a Markov chain with

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & & \end{pmatrix}$$

3 (c) (5 marks) Consider a Markov chain with state space $\mathcal{S} = \{1,2,3\}$ and transition probability matrix

$$P = \begin{pmatrix} 1/6 & 1/3 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove that $p_{12}^{(n)} \geq 1/3$ for all $n \geq 1$ (Hint: induction) and so conclude $\sum_{n=1}^{\infty} p_{12}^{(n)} = \infty$. Does this imply $f_{12} = 1$?

$p_{12}^{(1)} = 1/3$ and assume $p_{12}^{(n-1)} \geq 1/3$. Then

since $P^n = P\,P^{n-1} = P \begin{pmatrix} p_{11}^{(n-1)} & p_{12}^{(n-1)} & p_{13}^{(n-1)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

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we have that $p_{12}^{(n)} = \frac{1}{6} p_{12}^{(n-1)} + \frac{1}{3} \geq \frac{1}{3}$.

The fact that $\sum_{n=1}^{\infty} p_{12}^{(n)} = \infty$ does not

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imply that $f_{12} = 1$ since the chain is not irreducible.

3 (d) (5 marks) Consider a Markov chain with state space $\mathcal{S} = \{1, 2, 3\}$ and transition probability matrix

$$P = \begin{pmatrix} 1/6 & 1/3 & 1/2 \\ 1/6 & 2/3 & 1/6 \\ 2/3 & 0 & 1/3 \end{pmatrix}.$$

Determine a stationary distribution for this chain. Is the chain reversible with respect to this stationary distribution?

This transition probability matrix is doubly stochastic. Therefore the uniform distribution $\pi_i = 1/3$ for $i \in \{1,2,3\}$ is a stationary distribution.

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The chain is not reversible wrt π because

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$\pi_1 P_{12} = \frac{1}{3} \cdot \frac{1}{3} \neq \pi_2 P_{21} = \frac{1}{3} \cdot \frac{1}{6}$.

3(e) (5 marks) For the Markov chain in 3(d) determine $\lim_{n\to\infty} p_{ij}^{(n)}$ and fully justify this.

2 The chain is irreducible and since $p_{ii}^{(1)} > 0$ for every i the chain is aperiodic.

3 Therefore by the Markov Chain Convergence Theorem we have that $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j = \frac{1}{3}$.

4 (a) (5 marks) Suppose $(\Omega, \mathcal{A}, P)$ is a probability model with X and Y random variables. Using the general definition of conditional expecation prove that $E(XY \mid \mathcal{A}_X) = XY$ (Hint: use the uniqueness wp1 of the conditional expectation).

By the definition of conditional expectation

2 we have $E(H(X)Y) \overset{(*)}{=} E(H(X)E(Y|A_x))$ for

any r.v. $H(X)$. But since X and Y are stat. ind.

1 we have that $E(H(X)Y) = E(H(X))E(Y)$

and so we note that $E(Y|A_x) = Y$ satisfies

2 equation (*). Therefore, by the uniqueness

of $E(Y|A_x)$ we must have that $E(Y|A_x) = Y$.

so $E(XY|A_x) = X E(Y|A_x) = XY$.

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4 (b) (~~8~~ marks) Suppose $\{M_n : n \in \mathbb{N}_0\}$ is a martingale with respect to the stochastic process $\{X_n : n \in \mathbb{N}_0\}$, ~~namely~~, M_n is a function of $(X_0, X_1, \ldots, X_n)$, and also $\{B_n : n \in \mathbb{N}_0\}$ is a stochastic process where B_n is a function of $(X_0, X_1, \ldots, X_{n-1})$ s.t. $P(|B_n| < C) = 1$ for every n. If $Y_0 = 0$ and

$$Y_n = \sum_{i=1}^{n} B_i(M_i - M_{i-1}),$$

then prove that $\{Y_n : n \in \mathbb{N}_0\}$ is a martingale with respect to $\{X_n : n \in \mathbb{N}_0\}$.

We have $|Y_n| \le \sum_{i=1}^{n} |B_i|(|M_i| + |M_{i-1}|)$

$\le \sum_{i=1}^{n} C(|M_i| + |M_{i-1}|)$ so

5 $E|Y_n| \le \sum_{i=1}^{n} C(E|M_i| + E|M_{i-1}|) < \infty$

since $\{M_n : n \in \mathbb{N}_0\}$ is a martingale.

Now $Y_{n+1} = Y_n + B_{n+1}(M_{n+1} - M_n)$ so

$E(Y_{n+1} | X_0, \ldots, X_n) = E(Y_n | X_0, \ldots, X_n)$
$+ E(B_{n+1}(M_{n+1} - M_n) | X_0, \ldots, X_n)$

5 $= Y_n + B_{n+1}(E(M_{n+1} | X_0, \ldots, X_n) - E(M_n | X_0, \ldots, X_n))$

since Y_n, B_{n+1} are functions of $X_0, \ldots, X_n$

$= Y_n + B_{n+1}(M_n - M_n)$ since $\{M_n : n \in \mathbb{N}_0\}$ is a martingale

$= Y_n$

$\therefore \{Y_n : n \in \mathbb{N}_0\}$ is a martingale.

4 (c) (5 marks) Suppose for the processes defined in 4 (b) that T is a stopping time for the process $\{Y_n : n \in \mathbb{N}_0\}$.Establish that T also a stopping time for the process $\{X_n : n \in \mathbb{N}_0\}$.

We have that $\{T=n\} \in \mathcal{A}_{Y_0,\ldots,Y_n}$ but $\mathcal{A}_{Y_0,\ldots,Y_n} \subseteq \mathcal{A}_{X_0,\ldots,X_n}$ which proves T is also a stopping time for $\{X_n : n \in \mathbb{N}_0\}$.

4. (d) (5 marks) For the process $\{Y_n : n \in \mathbb{N}_0\}$ with stopping time T prove that the process $\{Y_{\min(n,T)} : n \in \mathbb{N}_0\}$ is also a martingale. (Hint: $Y_{\min(n+1,T)} = I_{\{T \le n\}} Y_{\min(n,T)} + I_{\{T>n\}} Y_{n+1}$.)

Using the hint

$$E(Y_{\min(n+1,T)} \mid X_0, \dots, X_n)$$

$$= E(I_{\{T \le n\}} Y_{\min(n,T)} \mid X_0, \dots, X_n) + E(I_{\{T>n\}} Y_{n+1} \mid X_0, \dots, X_n)$$

$$= I_{\{T \le n\}} Y_{\min(n,T)} + I_{\{T>n\}} E(Y_{n+1} \mid X_0, \dots, X_n)$$

since $I_{\{T \le n\}}$, $Y_{\min(n,T)}$, $I_{\{T>n\}} = 1 - I_{\{T \le n\}}$ are all functions of $X_0, \dots, X_n$

$$= I_{\{T \le n\}} Y_{\min(n,T)} + I_{\{T>n\}} Y_n$$ since $\{Y_n, n \in \mathbb{N}\}$ is a martingale

$$= Y_{\min(n,T)}$$ in both cases where $T \le n$ or $T > n$.

5. (a) (5 marks) If $\{B_t : t \geq 0\}$ is a Brownian motion, then determine $E((B_2 - B_{1/2})(B_2 - B_1)^2)$.

$$\mathbb{E}\left((B_2 - B_{1/2})(B_2 - B_1)^2\right)$$

$$= \mathbb{E}\left((B_2 - B_1 + B_1 - B_{1/2})(B_2 - B_1)^2\right)$$

$$= \mathbb{E}\left((B_2 - B_1)^3\right) + \mathbb{E}\left((B_1 - B_{1/2})(B_2 - B_1)^2\right)$$

$$= \mathbb{E}(Z^3) + \mathbb{E}(B_1 - B_{1/2})\,\mathbb{E}\left((B_2 - B_1)^2\right)$$

since $Z = B_2 - B_1 \sim N(0,1)$ independent of $B_1 - B_{1/2} \sim N(0, 1/2)$

$$= 0 + 0 = 0$$

since $\mathbb{E}(Z^3) = 0$, $\mathbb{E}(B_1 - B_{1/2}) = 0$.

5. (b) (5 marks) If $\{B_t : t \geq 0\}$ is a Brownian motion and $M_t = \max\{B_t : 0 \leq s \leq t\}$ then determine $E(M_t)$.

We have from Prop VI.8 that

3 M_t has density $f(m) = \sqrt{\frac{2}{\pi t}} \exp\{-\frac{m^2}{2t}\}$.

Then $E(M_t) = \sqrt{\frac{2}{\pi t}} \int_0^\infty m \exp\{-\frac{m^2}{2t}\} dm$

2 $= \sqrt{\frac{2}{\pi t}} \left(-t \exp\{-\frac{m^2}{2t}\} \Big|_0^\infty \right)$

$= \sqrt{\frac{2}{\pi t}}\, t = \sqrt{\frac{2t}{\pi}}$.

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6. (a) (marks) Suppose messages arrive according to a Poisson process with intenstity λ and a message is labelled as urgent with probability 1/3. If M_t is the total number of messages in $[0,t]$ and U_t is the number of messages labelled urgent in $[0,t]$, then compute $P(U_2 = 3)$, $P(U_2 = 3, M_5 = 4)$.

We have that $\{U_t : t \geq 0\}$ is a Poisson process with intensity $\lambda/3$ so

5 $P(U_2=3) = \frac{(2\lambda/3)^3}{3!} e^{-2\lambda/3}$. Also

$M_t - U_t$ = # of messages labelled not urgent is a Poisson process of intensity $2\lambda/3$ and it is independent of $\{U_t : t \geq 0\}$

Now $\{U_2=3, M_5=4\} = \{U_2=3, U_5-U_2=1, M_5-U_5=0\}$ ①

$\cup \{U_2=3, U_5-U_2=0, M_5-U_5=1\}$ ②

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Therefore, using independence of increments and the processes, and the disjointness of ① and ②,

$P(U_2=3, M_5=4) = P(①)+P(②) = \frac{(2\lambda/3)^3}{3!} e^{-2\lambda/3} \frac{(3\lambda/3)^1}{1!} e^{-3\lambda/3} e^{-5(2\lambda/3)}$

$+ \left(\frac{2\lambda/3}{3!}\right)^3 e^{-2\lambda/3} e^{-3\lambda/3} \left(\frac{5(2\lambda/3)}{1!}\right)^1 e^{-5(2\lambda/3)}$

6. (b) (5 marks) Suppose messages arrive from two independent sources according to a Poisson processes with intenstities λ_1 and λ_2 and U_t is the number of messages labelled urgent (with probability 1/3) in $[0, t]$. Compute $P(U_2 = 3)$.

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We have that the calls arrive according to a Poisson process with intensity $\lambda_1 + \lambda_2$. Therefore $\{U_t : t \geq 0\}$ is a Poisson process with intensity $(\lambda_1 + \lambda_2)/3$ which implies

$$P(U_2 = 3) = \frac{\left(2\frac{(\lambda_1+\lambda_2)}{3}\right)^3}{3!} e^{-2\frac{(\lambda_1+\lambda_2)}{3}}.$$