Solutions

STAC63 Midterm 2022

Any results established in the class or in the Exercises, appropriately referenced, can be used as part of solving these questions.

1(i). (10 marks) Suppose you need to approximate the following integral

$$I = \int_{\mathbb{R}^{10}} h(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x}$$

where $h(\mathbf{x}) = \sum_{i=1}^{10} x_i^3$, f is the $N_{10}(0, \Sigma)$ and

$$\Sigma = \begin{pmatrix} 1.0 & 0.5 & \dots & 0.5 \\ 0.5 & 1.0 & & \vdots \\ \vdots & & \ddots & 0.5 \\ 0.5 & \dots & 0.5 & 1.0 \end{pmatrix}.$$

Discuss fully how you would carry out such an approximation. You do not have to specify code but make sure you describe each step necessary to implement your approximation.

A sample $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ is generated from the $N_{10}(\underset{\sim}{0}, \Sigma)$ distribution. Then I is approximated by the average $I_n = \frac{1}{n}\sum_{i=1}^{n} h(\underset{\sim}{X}_i)$.

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To determine the accuracy of the approximation $s_n^2 = \frac{1}{n}\sum_{i=1}^{n} h^2(\underset{\sim}{X}_i) - I_n^2$ is computed and the interval

3 $\left(I_n - 3\frac{s_n}{\sqrt{n}}, I_n + 3\frac{s_n}{\sqrt{n}}\right)$ is computed.

To generate $\underset{\sim}{X} \sim N_{10}(\underset{\sim}{0}, \Sigma)$ the Cholesky factor T of Σ is computed and $Z_1, \ldots, Z_{10}$ is generated from the $N(0,1)$ distribution and put $\underset{\sim}{X} = T\underset{\sim}{Z}$ where $\underset{\sim}{Z} = (Z_1, \ldots, Z_n)'$.

3

1(ii). (10 marks) Suppose you want to generate a sample from a density $h(x) \propto |\cos(x)|\varphi(x)$ where φ is the standard normal density. Fully describe an algorithm to do this.

For this we use the rejection algorithm with $g(x) = \varphi(x)$ (the $N(0,1)$ density) and $c=1$

5 since $|\cos(x)| \leq 1$ which implies $h(x) \leq cg(x) = g(x)$ for all x. The actual algorithm proceeds as follows:

(1) generate $Y \sim N(0,1)$ and $U \sim \text{Uniform}(0,1)$

5 (2) if $U\varphi(Y) \leq |\cos(Y)|\varphi(Y)$ then return $X = Y$ otherwise go to 1

($U\varphi(Y) \leq |\cos(Y)|\varphi(Y)$ iff $U \leq |\cos(Y)|$)

2. For all the parts of question 2 justify any of your conclusions by citing relevant results from the notes.

2(i) (5 marks) Suppose that $X_1, \ldots, X_n$ is a sample (*i.i.d.*) from a distribution with mean μ. Specify what $\bar{X}^2 = \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2$ converges to and the mode of convergence.

1 By the SLLN we have $\bar{X} \xrightarrow{wp1} \mu$.

Then by the Continuous Mapping Thm ~~for~~

2 Convergence with probability 1 (Prop. IX.10)

we have $\bar{X}^2 \xrightarrow{wp1} \mu^2$ since

2 $g(x) = x^2$ is a continuous function.

2(ii) (5 marks) Suppose that $X_1, \ldots, X_n$ is a sample from a distribution with mean 0 and variance 1. What distribution does $n\bar{X}^2$ converge to and what is the mode of convergence?

2 By the CLT we have that $\sqrt{n}\bar{X} \xrightarrow{d} N(0,1)$. If $Z \sim N(0,1)$ then

1 $Z^2 \sim$ chi-squared(1). So by (Prop. II.1) the Continuous mapping theorem for convergence in distribution we have that

2 $n\bar{X}^2 = (\sqrt{n}\bar{X})^2 \xrightarrow{d}$ chi-squared(1) since $g(x) = x^2$ is a continuous function.

2(iii) (5 marks) Suppose that $X_1, \ldots, X_n$ is a sample from a distribution with mean 0 and $Y \sim N(1,2)$. What distribution does $\bar{X} + Y$ converge to?

2 We have that $\bar{X} \xrightarrow{p} 0$ by the WLLN.

Therefore by Prop VI.2 we have that

3 $\bar{X} + Y \xrightarrow{d} Y \sim N(1,2)$.

2(iv) (15 marks) Suppose that $X_1, \ldots, X_n$ is a sample from a distribution with mean 1 and variance 2. Determine a normal approximation to the distribution of $\bar{X}^3$ and show how this would be used to determine probabilities for this random variable.

5 By the CLT we have $\sqrt{\frac{n}{2}}(\bar{X}-1) \xrightarrow{d} N(0,1)$.
Also $g(x) = x^2$ has $g'(x) = 3x^2$ and $g(1) = 1$, $g'(1) = 3$. Therefore by the delta theorem

5 $$\sqrt{\frac{n}{2}}(\bar{X}^3 - 1) \xrightarrow{d} N(0, (g'(1))^2) = N(0,9)$$

so $\sqrt{\frac{n}{18}}(\bar{X}^3 - 1) \xrightarrow{d} N(0,1)$.

This implies that

$P(a \le \bar{X}^3 \le b)$

5 $= P\left(\sqrt{\frac{n}{18}}(a-1) \le \sqrt{\frac{n}{18}}(\bar{X}^3 - 1) \le \sqrt{\frac{n}{18}}(b-1)\right)$

$\approx \Phi\left(\sqrt{\frac{n}{18}}(b-1)\right) - \Phi\left(\sqrt{\frac{n}{18}}(a-1)\right)$

3(i) (5 marks) Suppose that $X_1, X_2 \ldots$ is an *i.i.d.* sequence from a distribution on $\mathbb{N}_0$ given by $P(X_1 = i) = p_i$ and $X_0 = 0$. Prove or disprove that $\{X_n : n \in \mathbb{N}_0\}$ is a Markov chain. Hint: what do you have to specify to determine a MC.

$\{X_n : n \in \mathbb{N}_0\}$ is a MC with initial distribution $\underline{v} = (1, 0, 0, \ldots)$ and transition probability matrix given by

$$P = \begin{pmatrix} p_0 & p_1 & p_2 & \cdots \\ p_0 & p_1 & p_2 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

3(ii) (5 marks) Consider a MC with state space $S = \{1, 2, 3, 4\}$ and transition probabilities

$$P = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 2/3 & 1/3 & 0 \\ 0 & 1/5 & 4/5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Which states are recurrent and which are transient?

We have that $1 \to 2, 3, 4$

$2 \to 3$

$3 \to 2$

2

So $2 \leftrightarrow 3$ and they are recurrent as is 4

but if we start in 1 the chain leaves

3

and never returns so 1 is transient.

3(iii) (5 marks) For the MC in 3(ii) compute $p_{12}^{(2)}$ and $p_{12}^{(3)}$.

$$P^2 = \begin{pmatrix} 1/16 & 1/16+2/12+1/20 & 1/16+1/12+1/5 & 1/16+1/4 \\ 0 & 4/9+1/16 & 2/9+4/15 & 0 \\ 0 & 2/15+4/25 & 1/15+16/25 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3

$$= \begin{pmatrix} \frac{1}{16} & \frac{67}{240} & \frac{166}{480} & \frac{5}{16} \\ 0 & \frac{69}{135} & \frac{56}{135} & 0 \\ 0 & \frac{22}{75} & \frac{53}{75} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\therefore\ p_{12}^{(2)} = \frac{67}{240}$ and

2 $\quad p_{12}^{(3)} = \frac{1}{4}\frac{67}{240} + \frac{1}{4}\frac{69}{135} + \frac{1}{4}\frac{22}{75} + \frac{1}{4}0$

no need to do the arithmetic

3(iv) (5 marks) For the MC in 3(ii) compute f_{23} and f_{14}.

$$f_{23} = p_{23} + p_{21} f_{13} + p_{22} f_{23} + p_{24} f_{43}$$

3

$$= \frac{1}{3} + 0 + \frac{2}{3} f_{23} + 0$$

$$\therefore \frac{1}{3} f_{23} = \frac{1}{3} \quad \text{or} \quad f_{23} = 1$$

2

$$f_{14} = p_{14} + p_{11} f_{14} + p_{12} f_{24} + p_{13} f_{34}$$

$$= \frac{1}{4} + \frac{1}{4} f_{14}$$ since $f_{24} = f_{34} = 0$

as once we are in $\{2, 3\}$

we never leave

$$\therefore f_{14} = \frac{1}{3}$$

3(v) (5 marks) For the MC in 3(ii) determine the closed classes and whether or not these are recurrent or transient.

The closed classes are $C_1 = \{2, 3\}$ and $C_2 = \{4\}$ and both are recurrent.

4. Consider a MC with state space $S = \{1, 2, 3\}$ and transition probabilities

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \\ 1/4 & 3/4 & 0 \end{pmatrix}.$$

4(i) (5 marks) Find a stationary distribution for this chain.

π must satisfy

$$\pi_1 \cdot 0 + \pi_2 \cdot \tfrac{1}{3} + \pi_3 \cdot \tfrac{1}{4} = \pi_1 \qquad \pi_1 = \frac{\pi_2}{3} + \frac{\pi_3}{4}$$

$$\pi_1 \cdot \tfrac{1}{2} + \pi_2 \cdot 0 + \pi_3 \cdot \tfrac{3}{4} = \pi_2 \quad \text{or} \quad \pi_1 = 2\pi_2 - \tfrac{3}{2}\pi_3$$

$$\pi_1 \cdot \tfrac{1}{2} + \pi_2 \cdot \tfrac{2}{3} + \pi_3 \cdot 0 = \pi_3$$

so $\frac{\pi_2}{3} + \frac{\pi_3}{4} = 2\pi_2 - \frac{3}{2}\pi_3$ or $\pi_2 = \frac{3}{5}\cdot\frac{7}{4}\pi_3 = \frac{21}{20}\pi_3$

$1:\ \pi_1 = \frac{\pi_2}{3} + \frac{1}{4}\cdot\frac{20}{21}\pi_2 = \left(\frac{1}{3} + \frac{5}{21}\right)\pi_2 = \frac{12}{21}\pi_2$

and $1 = \pi_1 + \pi_2 + \pi_3 = \frac{12}{21}\pi_2 + \pi_2 + \frac{20}{21}\pi_2 = \frac{53}{21}\pi_2$

so $\pi_2 = \frac{21}{53}$, $\pi_1 = \frac{12}{53}$, $\pi_3 = \frac{20}{53}$

4(ii) (5 marks) Determine (with explanation) whether or not $\lim_{n\to\infty} p_{13}^{(n)} = \pi_3$.

The chain is irreducible since

1 $1 \to 2$, $1 \to 3$, $2 \to 1$, $2 \to 3$, $3 \to 1$, $3 \to 2$ and so all states communicate.

Since $1 \to 2 \to 1$ and $1 \to 2 \to 3 \to 1$ we have

2 that $\{n : p_{11}^{(n)} > 0\} = \{2, 3, \ldots\}$ and so period(1) = 1. Since the chain is irreducible all states have period 1 and the chain is aperiodic.

2 Therefore by the MCCT, since the chain has stationary distribution we have

$$\lim_{n\to\infty} P_{13}^{(n)} = \pi_3 = \frac{20}{63}.$$

4(iii) (5 marks) Determine (with explanation) whether or not $f_{13} = 1$.

By Prop. III.9 (Finite State Space Thm)

2 the chain is recurrent. Then since

the chain is irreducible Lemma III.11

3 implies that $f_{13} = 1$.

4(iv) (5 marks) Determine (with explanation) whether or not $\sum_{n=1}^{\infty} p_{13}^{(n)} = \infty$.

By Prop. VI.8 (Cases Theorem) since

5 the chain is recurrent we have that

$$\sum_{n=1}^{\infty} p_{13}^{(n)} = \infty.$$

4(v) (5 marks) Determine whether or not this chain is time reversible.

2 $\pi_1 P_{12} = \frac{12}{53} \frac{1}{2} = \frac{6}{53}$ and $\pi_2 P_{21} = \frac{21}{53} \frac{1}{3} = \frac{7}{53}$

3 So $\pi_1 P_{12} \neq \pi_2 P_{21}$ and so the chain is not time reversible