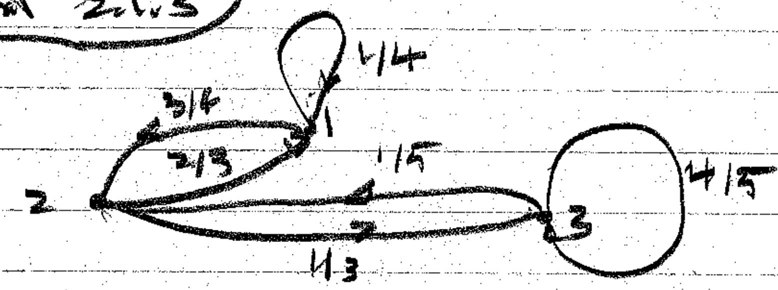
Lectures 4a, b, c, d Exercises

IV.1 Text 2.1.3

(a)



(b) $\frac{1}{4}\pi_1 + \frac{2}{3}\pi_2 = \pi_1 \quad \therefore \quad \pi_2 = \frac{9}{8}\pi_1$

$\frac{3}{4}\pi_1 + \quad \frac{1}{5}\pi_3 = \pi_2 \quad \therefore \pi_3 = \frac{15}{8}\pi_1$

$\frac{1}{3}\pi_2 + \frac{4}{5}\pi_3 = \pi_3$

then $1 = \pi_1 + \pi_2 + \pi_3 = \left(1 + \frac{9}{8} + \frac{15}{8}\right)\pi_1 = 4\pi_1$

$\therefore \pi_1 = \frac{1}{4} \quad \pi_2 = \frac{9}{32} \quad \pi_3 = \frac{15}{32}$

IV.2 Text 2.2.4

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \pi = (1/3, 1/3, 1/3)$$

(a) P is doubly stochastic so uniform is the stationary distribution which implies π is stationary

(b) $\pi_1 p_{12} = \frac{1}{3} \neq \pi_2 p_{21} = 0$

$\therefore$ detailed balance is not satisfied.

IV.3 Text 2.3.5

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$\{n : p_{11}^{(n)} > 0\} = \{3, 6, 9, \ldots\}$ as $1 \to 2 \to 3 \to 1$ is the only possible path from 1 to 1.
Since the chain is irreducible all states have the same period, namely, 3.

IV.4 Text 2.4.15

(a) $p_{13}^{(2)} = (1/3, 0, 2/3) \begin{pmatrix} 2/3 \\ 1/2 \\ 1/15 \end{pmatrix} = \frac{2}{9} + 0 + \frac{2}{45}$

$$= \frac{10}{45} + \frac{6}{45} = \frac{16}{45}$$

(b) State 2 is transient because the chain leaves state 2 with probability 2/3 and $C = \{1, 3\}$ is closed. The chain with state space is irreducible and so is recurrent since the state space is finite

(c) $f_{23} = p_{23} + p_{21} f_{13} + p_{22} f_{23}$

$= 1/2 + 1/6 f_{13} + 1/3 f_{23}$

and since C is recurrent $f_{13} = 1$

$= 1/2 + 1/6 + 1/3 f_{23}$

$\therefore f_{23} = \frac{3}{2}\left(\frac{2}{3}\right) = 1$

(d) ① $\frac{1}{3}\pi_1 + \frac{1}{6}\pi_2 + \frac{4}{5}\pi_3 = \pi_1$

② $\frac{1}{3}\pi_2 = \pi_2$ $\therefore \pi_2 = 0$

③ $\frac{2}{3}\pi_1 + \quad + \frac{1}{5}\pi_3 = \pi_3$

from ① $\frac{4}{5}\pi_3 = \frac{2}{3}\pi_1$ $\therefore \pi_3 = \frac{10}{12}\pi_1 = \frac{5}{6}\pi_1$

$\therefore 1 = \pi_1 + \pi_2 + \pi_3 = \left(1 + \frac{5}{6}\right)\pi_1 = \frac{11}{6}\pi_1$

$\therefore \pi_1 = \frac{6}{11}$, $\pi_2 = 0$, $\pi_3 = \frac{5}{11}$

(e) $\pi_1 P_{12} = 0 = \pi_2 P_{21}$

$\pi_1 P_{13} = \frac{6}{11}\frac{2}{3} = \frac{4}{11} = \pi_3 P_{31} = \frac{5}{11}\frac{4}{5} = \frac{4}{11}$

$\pi_2 P_{23} = 0 = \pi_3 P_{32}$

$\therefore$ the chain is time reversible.

(f) Since $C = \{1, 3\}$ is closed and irreducible and $P_{11} = 1/3$, so period(1) = period(3) = 1 the MCCT implies $\lim_{n\to\infty} P_{11}^{(n)} = \pi_1$, $\lim_{n\to\infty} P_{13}^{(n)} = \pi_3$

Also $1 = \lim_{n\to\infty} 1 = \lim_{n\to\infty}\left(P_{11}^{(n)} + P_{12}^{(n)} + P_{13}^{(n)}\right)$

$= \pi_1 + \lim_{n\to\infty} P_{12}^{(n)} + \pi_3 = 1 + \lim_{n\to\infty} P_{12}^{(n)}$

$\therefore \lim_{n\to\infty} P_{12}^{(n)} = 0$.

1 V. 5 Text 2.5.4

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

(a) $\left(a, \frac{1-a}{2}, \frac{1-a}{2}\right) P = \left(a, \frac{1-a}{2}, \frac{1-a}{2}\right)$

and so $\left(a, \frac{1-a}{2}, \frac{1-a}{2}\right)$ is a stationary distribution for every $a \in [0,1]$

$\therefore$ the chain has infinitely many stationary distributions

(b) Note that the chain is not irreducible so the MCCT cannot be applied and this was required to prove the uniqueness (note this chain is aperiodic).

IV.6 Text 2.6.3

We use the Metropolis algorithm to construct the chain. So we have

$$p_{ij} = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \frac{1}{2}\min\{1, (\frac{2}{3^{i+1}})/(\frac{2}{3^i})\} = \frac{1}{6} & \text{if } j = i+1 \\ \frac{1}{2}\min\{1, (\frac{2}{3^{i-1}})/(\frac{2}{3^i})\} = \frac{1}{2} & \text{if } i \neq 1 \text{ and } j = i-1 \\ 0 & \text{if } i=1, j=0 \\ 1-\frac{1}{6}-\frac{1}{2} = \frac{1}{3} & \text{if } i=j \end{cases}$$

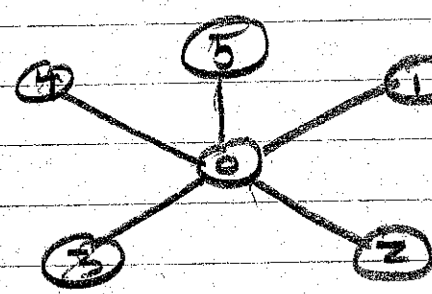
Then the Metropolis chain is irreducible since for $i<j$ $p_{i,i+1}p_{i+1,i+2}\cdots p_{j-1,j} > 0$ and there is a positive probability that $U_1 < 1/6, \ldots, U_{n-1} < 1/6$ so $P_i(X_1 = i+1, X_2 = i+2, \ldots, X_{j-i} = j) > 0$.
Also there is a positive probability of staying in a given state so the chain is aperiodic. Then the MCMC Convergence Thm applies.

IV.7 Text 2.6.4

This is just like Metropolis-Hastings but the proposal distribution also is symmetric, i.e. $q(i,j) = q(j,i)$ and so all the results proved in class for Metropolis-Hastings apply in this case. For example

$$p_{ij} = \begin{cases} q(i,j)\min\left\{1, \frac{\pi_j}{\pi_i}\right\} & \text{when } i \neq j \\ 1 - \sum_{k \neq i} p_{ik} & \text{when } i = j \end{cases}$$

IV.8 Text 2.7.10



a connected graph

(a) Clearly the random walk is irreducible because there is a path having positive probability between any two vertices as can be seen from the transition matrix

$$P = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}$$

(b) period (1) = gcd {2, 4, 6, ...} = 2
and by irreducibility all states have the same period.

(c) $\lim_{n\to\infty} P_{ii}^{(n)}$ does not exist, as counting possible paths

$P_{00}^{(2n)} = 5^n \left(\frac{1}{5}\right)^n = 1$, $P_{00}^{(2n+1)} = 0$

$P_{0i}^{(2n)} = 0$ for $i = 1, \dots, 5$, $P_{0i}^{(2n+1)} = 5^{-n}\left(\frac{1}{5}\right)^{n+1} = \frac{1}{5}$

$P_{i0}^{(2n)} = 0$ for $i = 1, \dots, 5$, $P_{i0}^{(2n+1)} = 5^{n}\left(\frac{1}{5}\right)^{n+1} = \frac{1}{5}$

$P_{ii}^{(2n)} = 5^{n-1}\left(\frac{1}{5}\right)^n = \frac{1}{5}$ $i \in \{1, 2, 3, 4, 5\}$, $P_{ii}^{(2n+1)} = 0$

$P_{ij}^{(2n)} = 5^{n-1}\left(\frac{1}{5}\right)^n = \frac{1}{5}$ $i, j \in \{1, 2, 3, 4, 5\}$ and $i \neq j$
and $P_{ij}^{(2n+1)} = 0$

$\therefore P_{ii}^{(n)}$ oscillates and does not converge

(d) $\lim_{n\to\infty} \frac{1}{2}\left(P_{ij}^{(n)} + P_{ij}^{(n+1)}\right)$

$$= \begin{cases} 1/2 & i = j = 0 \\ 1/10 & \text{otherwise} \end{cases}$$

and note.

$\left(\frac{1}{2}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}\right) P$

$= \left(\frac{1}{2}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}\right)$ which is the stationary distribution

10.9 Text 2.7.11

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

(a) This chain is irreducible for the same reason as in 2.7.10.

(b) period(1) $= \gcd\{1, 2, 3, \dots\} = 1$ and so the chain is aperiodic.

(c) and (d) It is easy to see that

$$\pi = \left(\frac{6}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}\right)$$

is a stationary distribution for the chain.

Therefore $$\lim_{n\to\infty} p_{ij} = \begin{cases} 6/11 & j=1 \\ 1/11 & \text{otherwise} \end{cases}$$

by the MCCT.