Solutions

STAC62 Test 2 2021

Any results established in the class or in the Exercises can be used as part of solving these questions.

1(i). (10 marks) If $\Omega = \{1, 2, 3, 4\}$, $\mathcal{A} = \{\phi, \{1\}, \{2\}, \{1, 2\}, \{3, 4\}, \{2, 3, 4\}, \{1, 3, 4\}, \Omega\}$, with $P(A) = \#(A)/4$ and $X : \Omega \to R^1$ is given by

$$X(1) = 0, X(2) = -1, X(3) = 1, X(4) = 1.$$

Justify that X is a random variable. Is X a discrete or absolutely continuous random variable? Determine the distribution function of X.

Since $X^{-1}(-\infty, a] = \begin{cases} \phi & \text{if } a < -1 \\ \{2\} & \text{if } -1 \le a < 0 \\ \{1,2\} & \text{if } 0 \le a < 1 \\ \{1,2,3,4\} & \text{if } a \geq 1 \end{cases} \in \mathcal{A}$

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this establishes that X is a r.v.

3 Since X takes only finitely many values then X is a discrete r.v.

$F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{4} & -1 \le x < 0 \\ \frac{1}{2} & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$

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1(ii). (5 marks) For X in (i) determine its probability function or density function, whichever is appropriate.

$$P_X(x) = P(X^{-1}\{x\})$$

$$= \begin{cases} 0 & x \notin \{-1, 0, 1\} \\ \frac{1}{4} & x = -1 \\ \frac{1}{4} & x = 0 \\ \frac{1}{2} & x = 1 \end{cases}$$

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1(iii). (10 marks) Justify that X in (i) is a simple function. Write X in canonical form and determine $E(X)$.

3 Since X takes only finitely many values it is a simple function.

3 $$X = (-1) I_{X^{-1}\{-1\}} + 0 I_{X^{-1}\{0\}} + 1 I_{X^{-1}\{1\}}$$

$$= - I_{\{2\}} + I_{\{3,4\}}$$

$$E(X) = E(-I_{\{2\}} + I_{\{3,4\}})$$

4 $$= -P(\{2\}) + P(\{3,4\})$$

$$= -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$

2. Consider probability model $(\mathbb{R}^1, B^1, P_X)$ where P_X is absolutely continuous with density

$$f_X(x) = \begin{cases} 3x^2 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

2(i) (5 marks) Prove that f_X is a density function.

2 Clearly $f_X(x) \geq 0 \quad \forall x \in \mathbb{R}^1$.

3 Also $\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 3x^2\,dx$

$= x^3 \Big|_0^1 = 1 - 0 = 1.$

Therefore f_X is a density function.

2(ii) (5 marks) Determine the distribution function F_X for X.

$$F_X(x) = \int_{-\infty}^{x} f_X(z)\,dz$$

$$= \begin{cases} 0 & , x < 0 \\ \int_0^x 3z^2\,dz & , 0 \leq x < 1 \\ 1 & , x \geq 1 \end{cases}$$

$$= \begin{cases} 0 & x < 0 \\ z^3 \Big|_0^x & 0 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases}$$

$$= \begin{cases} 0 & , x < 0 \\ x^3 & , 0 \leq x \leq 1 \\ 1 & , x \geq 1 \end{cases}$$

2(iii) (5 marks) Suppose that $Y = X^2$. Determine the density function for the distribution of Y.

Let $T : [0,1] \to \mathbb{R}^1$ be given by

$y = T(x) = x^2$ so T is 1-1 and $T^{-1}(y) = \sqrt{y}$.

Now $f_Y(y) \overset{2}{=} f_X(T^{-1}(y)) J_T(T^{-1}(y))$

where $J_T(x) = \left| \det\left(\frac{\partial T(x)}{\partial x} \right) \right|^{-1} = |2x|^{-1} = \frac{1}{2x}$

so $J_T(T^{-1}(y)) = \frac{1}{2\sqrt{y}}$. Therefore,

$$f_Y(y) = \begin{cases} 0 & y \notin [0,1] \\ 3(T^{-1}(y))^2 \frac{1}{2\sqrt{y}} & 0 \le y \le 1 \end{cases}$$

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$$= \begin{cases} 0 & y \notin [0,1] \\ \frac{3}{2}\sqrt{y} & 0 \le y \le 1 \end{cases}$$

2(iv) (5 marks) Suppose that $Y = 2I_{(0,1/4)}(X)+3I_{(1/2,2/3)}(X)$. Determine $E(Y)$.

3 Since Y is a simple function

$$E(Y) = 2P\left(0 < X < \tfrac{1}{4}\right) + 3P\left(\tfrac{1}{2} < X < \tfrac{2}{3}\right)$$

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$$= 2\left(\tfrac{1}{4}\right)^3 + 3\left(\left(\tfrac{2}{3}\right)^3 - \left(\tfrac{1}{2}\right)^3\right)$$

$$= \frac{2}{64} + 3\left(\frac{8}{27} - \frac{1}{8}\right)$$

2(v) (5 marks) Suppose that $Y = F_X(X)$. Determine the distribution of Y.

We have $0 \le Y \le 1$ and for $0 \le y \le 1$

$$F_Y(y) = P(Y \le y) = P(F_X(X) \le y)$$
$$= P(X^3 \le y) = P(X \le y^{1/3})$$
$$= F_X(y^{1/3}) = y$$

Therefore $f_Y(y) = 1$ for $0 \le y \le 1$ and we conclude that $Y \sim \text{Uniform}(0,1)$.

3(i). (5 marks) Suppose there are two random variables $X_1$and X_2 and we are told that

$$X_1 \sim N(0,1), \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\right).$$

Does this define a stochastic process $\{(t, X_t) : t \in \{1,2\}\}$? Justify your answer.

This defines a stochastic process because when (X_1, X_2) has the joint distribution provided we proved in class that $X_1 \sim N(0,1)$. Since this agrees with the stated distribution for X_1 the Kolmogorov Consistency Theorem says that this is a valid distribution of a stochastic process.

3(ii). (5 marks) Suppose that

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\right).$$

Determine the conditional distribution of X_2 given that $X_1 = 2$.

By Corollary $\underline{\text{IV}}$.8.2

$$X_2 | X_1 = 2 \sim N\left(1 + (-1)'(1)^{-1}(2-1), 2 - (-1)'(1)^{-1}(-1)\right)$$

$$= N(0, 1).$$

3(iii). (15 marks) For $\mathbf{X}$ in (ii) and $\mathbf{Y} = \mathbf{a}+B\mathbf{X}$ with

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

determine the distribution of $\mathbf{Y}$. Determine the marginal distributions of Y_1 and Y_2 and justify whether or not Y_1 and Y_2 are statistically independent.

From results proved in class we have

$$\underset{\sim}{Y} \sim N_2(\underset{\sim}{a} + B\underset{\sim}{\mu}_X, B\Sigma_X B')$$

6 $$= N_2\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$$

$$= N_2\left(\begin{pmatrix} 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

4 Therefore $Y_1 \sim N(3,1)$, $Y_2 \sim N(5,1)$

5 The joint density of $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ is

$$(2\pi)^{-1}(\det I)^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}\begin{pmatrix} y_1-3 \\ y_2-5 \end{pmatrix}' I \begin{pmatrix} y_1-3 \\ y_2-5 \end{pmatrix}\right\}$$

$$= (2\pi)^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}(y_1-3)^2\right\} (2\pi)^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}(y_2-5)^2\right\}$$

and so Y_1 and Y_2 are stat. ind.

4(i). (5 marks) Suppose that

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_{31} \end{pmatrix} \sim \text{multinomial}(5, 1/2, 1/3, 1/6).$$

Determine the distribution of $Y = X_1 + X_2$ and justify your answer.

Since X_i = the number of responses that fall in category i in an i.i.d. sample of n

then $X_1 + X_2$ = the number of responses falling in category 1 or 2 in an i.i.d. sample of n

Since a response falls in category 1 or category 2 with probability p_1+p_2 this proves that

$Y \sim$ multinomial(n, p_1+p_2, p_3)

= binomial(n, p_1+p_2)

4(ii). (5 marks) Suppose X and Y are random variables with respect to the probability model $(\Omega, \mathcal{A}, P)$. Prove that $W = \max(X, Y)$ is a random variable (hint: consider the inverse images of Borel sets of the form $(-\infty, w]$).

We have that $W^{-1}(-\infty, w]$

$= \{\omega : W(\omega) \le w\} = \{\omega : X(\omega) \le w \text{ and } Y(\omega) \le w\}$

$= \{\omega : X(\omega) \le w\} \cap \{\omega : Y(\omega) \le w\}$

$\in \mathcal{A}$ since both of those sets are in $\mathcal{A}$ which is a σ-algebra and so closed under finite intersections

4(iii). (10 marks) Suppose that X and Y in (ii) are statistically independent with common cdf F. Determine the cdf of W.

$$F_W(w) = P(W \le w)$$
$$= P(X \le w \text{ and } Y \le w)$$
$$= P(X \le w)\,P(Y \le w)$$

since X and Y are stat. ind.

$$= F(w)F(w) = F^2(w)$$

4(iv). (5 marks) Suppose that X is a random variable and $X_n = X/n$. Establish that X_n is a random variable for each n and that X_n converges to 0 with probability 1.

We have that $\{\omega : X_n(\omega) \le a\}$

$= \{\omega : X(\omega) \le na\} = X^{-1}(-\infty, na] \in \mathcal{A}$

and since this holds for every a

this proves that X_n is a r.v.

Now $X(\omega) \in \mathbb{R}$ and so $X(\omega)/n \to 0$

as $n \to \infty$. Therefore $\{\omega : X_n(\omega) \to 0\}$

$= \Omega$ and $P(\Omega) = 1$ which establishes

the result.

4(v). (5 marks) Suppose that X is a random variable and $X_n = XI_{(-n,n)}(X)$. Establish that X_n is a random variable for each n and that X_n converges to X with probability 1.

Note that if $X(\omega) \in (-n,n)$ then $X_n(\omega) = X(\omega)$ and for any ω $\exists\, n_0$ s.t. $X(\omega) \in (-n,n)$ $\forall\, n \geq n_0$ since the intervals $(-n,n)$ are increasing. Therefore $\{\omega : X_n(\omega) \to X(\omega) \text{ as } n \to \infty\} = \Omega$ which has probability 1.