STAC62 Test 1 2021

Any results established in the class or in the Exercises can be used as part of solving these questions.

1(i). (5 marks) If $\Omega = \{1, 2, 3, 4\}$, is the set $\mathcal{A} = \{\phi, \{1\}, \{2\}, \{2, 3, 4\}, \{1, 3, 4\}, \Omega\}$ a σ-algebra on Ω? Justify your answer.

No it is not a σ-algebra because

$\{1,2\} = \{1\} \cup \{2\} \notin \mathcal{A}$.

1(ii). (5 marks) If $\Omega = \{1, 2, 3, 4\}$, what is the σ-algebra generated by $\mathcal{C} = \{\{1, 2, 3\}, \{2, 3, 4\}\}$?

$$\mathcal{A}(\mathcal{C}) = \{\phi, \{1\}, \{4\}, \{1,4\}, \{2,3\}, \{1,2,3\}, \{2,3,4\}, \Omega\}$$

1(iii). (5 marks) Suppose $\mathbb{Q}$ is the set of rational numbers. Establish that $\mathbb{Q}$ is a Borel subset of $\mathbb{R}^1$.

As was argued in class $\mathbb{Q}$ is a

2 countable set. Also in the Exercises it was established that $\{b\} \in \mathcal{B}^1$ for every $b \in \mathbb{R}^1$. Therefore

3 $$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$$ is a Borel set

as it is the countable union of Borel sets.

1(iv). (10 marks) Prove that the lines given by $\{(x,y) : y = c\}$ and $\{(x,y) : x = d\}$, for constants c and d, are Borel subsets of $\mathbb{R}^2$.

We have that $\{(x,y) : y = c\}$

2 $= \mathbb{R}^1 \times \{c\}$ and $\mathbb{R}^1 \in \mathcal{B}^1$, $\{c\} \in \mathcal{B}^1$.

By a result in the notes if $B_1, B_2 \in \mathcal{B}^1$ then $B_1 \times B_2 \in \mathcal{B}^2$. Therefore

3 $\{(x,y) : y = c\} \in \mathcal{B}^2$ since $\mathbb{R}^1, \{b\} \in \mathcal{B}^1$.

Similarly $\{(x,y) : x = d\} = \{d\} \times \mathbb{R}^1$

5 is a Borel set

2. Consider probability model $(\mathbb{R}^1, B^1, P)$ where P is the uniform probability measure on $[1, 4]$.
2(i) (5 marks) What is $P([0, 3])$?

$$P([0,3]) = P([0,1) \cup [1,3])$$

$$= P([0,1)) + P([1,3])$$ as these sets are disjoint

$$= 0 + \frac{3-1}{4-1} = \frac{2}{3}.$$

2(ii) (10 marks) Let $B_n = [2+1/n, 2.5+2/n]$ for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty} B_n = (2, 2.5]$. Prove that indeed $\limsup_{n\to\infty} B_n = (2, 2.5]$.

Let $x \in \limsup_n B_n = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} B_i$. Then $\forall n$ $x \in \bigcup_{i=n}^{\infty} B_i$ and $\exists\, i_n \geq n$ s.t. $x \in B_{i_n} = [2+1/i_n, 2.5+2/i_n]$. This establishes that $x > 2$. Suppose $x > 2.5$. Then $\exists\, n_0$ s.t. $\forall n \geq n_0$ $2.5+2/n < x$ and so $\nexists\, i_n \geq n$ s.t. $x \in B_{i_n}$ which is a contradiction. Therefore $x \in (2, 2.5]$ and $\limsup_n B_n \subseteq (2, 2.5]$.

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Now suppose $x \in (2, 2.5]$. Then for any n $\exists\, i > n$ s.t. $2+1/i < x \leq 2.5+2/i$ and so $x \in \bigcup_{i=n}^{\infty} B_i$ $\forall n$ which implies $x \in \limsup_n B_n$. Therefore $(2, 2.5] \subseteq \limsup_n B_n$ and we have proved $\limsup_n B_n = (2, 2.5]$.

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2(iii) (5 marks) Calculate $P(B_n)$ and show that this converges to $P((2, 2.5])$.

$$P(B_n) = P([2+1/n, 2.5+2/n])$$

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$$= \begin{cases} P([2+1/n, 4]), & n=1 \\ P([2+1/n, 2.5+2/n]), & n>1 \end{cases}$$

3

$$= \begin{cases} \dfrac{4-2-1/n}{3} = \dfrac{2-1/n}{3}, & n=1 \\ \dfrac{2.5+2/n-2-1/n}{3} = \dfrac{0.5+1/n}{3}, & n>1 \end{cases}$$

$$\to \frac{1}{6} = \frac{2.5-2}{3} = P([2, 2.5])$$

2(iv) (5 marks) Justify why $P([a,b]) = P((a,b))$ for all $a \leq b$.

$P([a,b]) = P(\{a\} \cup (a,b) \cup \{b\})$

and $\{a\}, (a,b), \{b\}$ are disjoint Borel sets with

3 $P(\{a\}) = \begin{cases} 0 & a<1 \text{ or } a>4 \\ \frac{a-a}{3} = 0 & \text{when } 1 \leq a \leq 4 \end{cases}$

$= 0$

and similarly $P(\{b\}) = 0$.

Therefore $P([a,b])$

2 $= P(\{a\}) + P((a,b)) + P(\{b\})$

$= P((a,b))$

3(i). (5 marks) Suppose $\Omega = \{1, 2, 3, 4\}$ and $X : \Omega \to \mathbb{R}^1$ is given by $X(1) = 0, X(2) = 1, X(3) = -1, X(4) = 0$. Determine $X^{-1}\{-1, 0\}$.

$X^{-1}\{-1,0\} = \{1,3,4\}$

3(ii). (5 marks) For Ω in (i) let $\mathcal{A} = \{\phi, \{1,2\}, \{3,4\}, \Omega\}$. Prove that $\mathcal{A}$ is a σ-algebra on Ω and establish whether or not X is a random variable with respect to $\mathcal{A}$.

We have $\{1,2\} \cup \{3,4\} = \Omega$ and $\{1,2\}^c = \{3,4\}$

$\{3,4\}^c = \{1,2\}$ and complementation and

unions with ϕ and Ω are obvious.

2 Therefore, $\mathcal{A}$ is a σ-algebra as it is closed under unions and complementation.

Also $X^{-1}B = \begin{cases} \phi & \text{if } -1, 0, 1 \notin B \\ \{3\} & \text{if } -1 \in B,\ 0, 1 \in B^c \end{cases}$

3 and we don't have to proceed further since $\{3\} \notin \mathcal{A}$ establishes that X is not a random variable.

3(iii). (10 marks) Suppose P is the uniform probability measure on Ω in (i) with $A = 2^{\Omega}$. Determine P_X.

$$P_X(\{0\}) = P(\{1,4\}) = \frac{2}{4} = \frac{1}{2}$$

$$P_X(\{-1\}) = P(\{3\}) = \frac{1}{4}$$

5 $$P_X(\{1\}) = P(\{2\}) = \frac{1}{4}$$

$\therefore$ for any $B \in \mathcal{B}^1$

$$P_X(B) = \begin{cases} 0 & \text{if } 0,-1,1 \notin B \\ 1/2 & \text{if } 0 \in B,\ -1,1 \notin B \\ 1/4 & \text{if } 0 \notin B,\ -1 \in B,\ 1 \notin B \\ 1/4 & \text{if } 0 \notin B,\ -1 \notin B,\ 1 \in B \\ 3/4 & \text{if } 0,-1 \in B,\ 1 \notin B \\ 3/4 & \text{if } 0,1 \in B,\ -1 \notin B \\ 1/2 & \text{if } 0 \notin B,\ 1,-1 \in B \\ 1 & \text{otherwise} \end{cases}$$

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3(iv). (5 marks) Is P_X in (iii) a discrete or absolutely continuous probability measure? Justify your conclusion.

It is a discrete probability distribution because the probability function $p_X(x) = 0$ if $x \neq 2,3$

$= \frac{1}{2}$ if $x = 2$ or $x = 3$

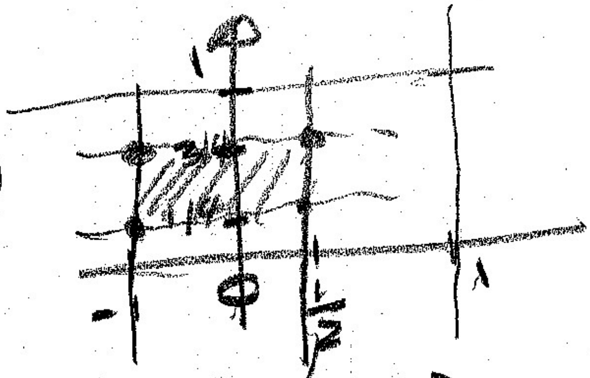
satisfies $p_X(x) \geq 0$; $\sum_x p_X(x) = 1$.

4. Suppose $(R^2, \mathcal{B}^2, P)$ is a probability model where P is a probability measure with $P([0,1]^2) = 1$.

4(i). (5 marks) Suppose that P has cdf F given by

$$F(x,y) = \begin{cases} 0 & \text{if} \quad x < 0 \text{ or } y < 0, \\ xy^2 & \text{if} \quad 0 < x < 1 \text{ and } 0 < y < 1, \\ x & \text{if} \quad 0 < x < 1 \text{ and } y \geq 1, \\ y^2 & \text{if} \quad x \geq 1 \text{ and } 0 < y < 1, \\ 1 & \text{if} \quad x \geq 1 \text{ and } y \geq 1. \end{cases}$$

Determine $P([-1,1/2] \times (1/4,3/4))$.



$$P([-1,1/2] \times (1/4,3/4))$$

$$= F(1/2, 3/4)$$

3 $$- F(1/2, 1/4) - F(-1, 3/4) + F(-1, 1/4)$$

2 $$= \tfrac{1}{2}\left(\tfrac{3}{4}\right)^2 - \tfrac{1}{2}\left(\tfrac{1}{4}\right)^2 - 0 + 0 = \tfrac{1}{2}\left(\tfrac{9}{16} - \tfrac{1}{16}\right) = \tfrac{1}{4}$$

4(ii). (5 marks) Consider the events $A = \{(x,y) : x \in [-1, 1/2]\}$ and $C = \{(x,y) : y \in [1/4, 3/4]\}$. Determine whether or not A and C are statistically independent.

$$P(A) = P([-1, 1/2] \times \mathbb{R})$$

$$= F(1/2, \infty) - F(-1, \infty)$$

$$= \frac{1}{2}$$

2

$$P(C) = P(\mathbb{R} \times [1/4, 3/4])$$

$$= F(\infty, 3/4) - F(\infty, 1/4)$$

$$= \left(\frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2 = \frac{1}{2}.$$

$$\therefore P(A \wedge C) = P([-1, 1/2] \times [1/4, 3/4])$$

3

$$= \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A)\,P(C)$$

and this proves A and C are stat. independent.

4(iii). (5 marks) Determine whether or not P is a continuous probability measure and justify this. If P is absolutely continuous determine the probability density function for P.

$$P(\{(x,y)\}) = \lim_{\delta \downarrow 0} P((x-\delta, x] \times (y-\delta, y]) = 0 \quad \forall (x,y)$$

2 because F is a continuous function so P is continuous. Also

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = \begin{cases} 0 & \text{if } x \notin [0,1] \text{ or } y \notin [0,1] \\ 2y & \text{if } (x,y) \in [0,1]^2 \end{cases}$$

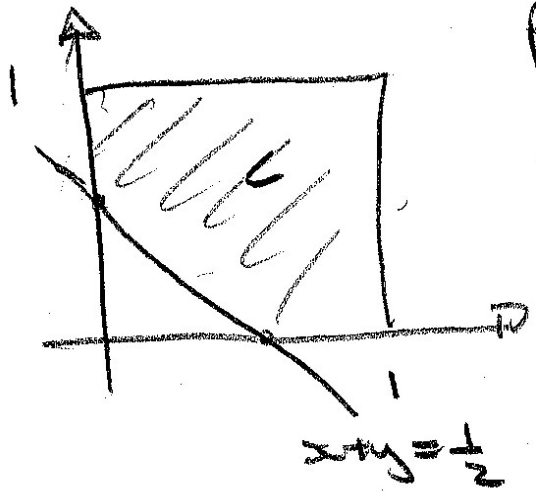
3 which is nonnegative and

$$\int_{\mathbb{R}^2} f(x,y)\,dx\,dy = \int_0^1 \int_0^1 2y\,dx\,dy$$

$$= \int_0^1 dx \int_0^1 2y\,dy = y^2 \Big|_0^1 = 1$$

so P is absolutely continuous with this density.

4(iv). (10 marks) Consider the events $A = \{(x,y) : x \in [-1, 1/2]\}$ and $C = \{(x,y) : x + y \geq 1/2\}$ and compute $P(A\,|\,C)$. Are A and C are statistically independent?

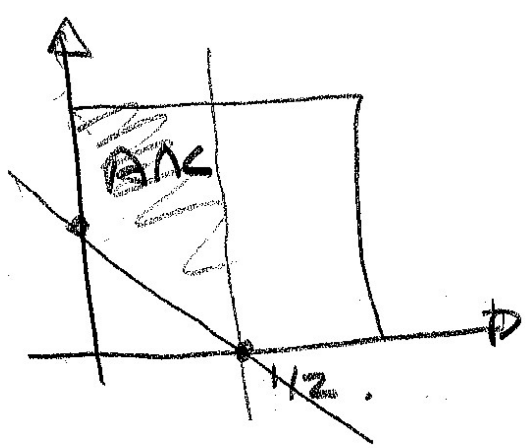


$$P(C^c) = \int_0^{1/2} \int_0^{1/2-y} 2y \, dx\, dy$$

$$= \int_0^{1/2} 2y\left(\tfrac{1}{2} - y\right) dy$$

$$= \frac{y^2}{2} - \frac{2y^3}{3} \Big|_0^{1/2} = \frac{1}{8} - \frac{1}{12} = \frac{1}{24}$$

$$\therefore P(C) = 1 - P(C^c) = \frac{23}{24}$$



$$P(A) = F\left(\tfrac{1}{2}, \infty\right) = \frac{1}{2}$$

$$P(A \cap C) = P(A \setminus (A \cap C^c))$$

$$= P(A) - P(A \cap C^c)$$

since $A \cap C^c \subseteq A$

$$= \frac{1}{2} - \frac{1}{24} = \frac{11}{24}$$

$$\therefore P(A\,|\,C) = \frac{P(A \cap C)}{P(C)} = \frac{11/24}{23/24} = \frac{11}{23} \neq \frac{1}{2}$$

and so A and C are not stat. ind.