STAC62 Test 2 2023

Any results established in the class or in the Exercises can be used as part of solving these questions.

Solutions

1(a). (10 marks) Suppose that $(\Omega, \mathcal{A}, P)$ where $\Omega = \mathbb{R}^1, \mathcal{A} = \mathcal{B}^1$ and P is the uniform probability measure on $[0,1]$. If $X : \Omega \to \mathbb{R}^1$ is given by $X(\omega) = \omega^3$, then prove that X is a random variable.

We need to show that $X^{-1}B \in \mathcal{B}^1$ for every $B \in \mathcal{B}^1$. Based on results in class this only requires that we show that $X^{-1}(-\infty, c] \in \mathcal{B}^1$ for every $c \in \mathbb{R}^1$. Now

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$$X^{-1}(-\infty, c] = \begin{cases} (-\infty, -|c|^{1/3}] & \text{when } c \leq 0 \\ (-\infty, c^{1/3}] & \text{when } c > 0 \end{cases}$$

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and in both cases, this set is in $\mathcal{B}^1$.

Therefore X is a r.v.

1(b). (10 marks) For X in (a) determine its distribution and density functions.

We have that $F_X(x) = P(X \leq x)$

$$= P(\{\omega : X(\omega) \leq x\})$$

$$= P(\{\omega : \omega^3 \leq x\})$$

$$= \begin{cases} P(\{\omega : |\omega| \geq |-|x|^{1/3}\}) & \text{when } x < 0 \\ P(\{\omega : \omega \leq x^{1/3}\}) & \text{when } x \geq 0 \end{cases}$$

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$$= \begin{cases} 0 & x \leq 0 \\ x^{1/3} & 0 < x < 1 \\ 1 & 1 \leq 1 \end{cases}$$

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$$\therefore f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} 0 & x < 0 \\ \frac{1}{3} x^{-2/3} & 0 < x < 1 \\ 0 & 1 \leq x \end{cases}$$

1(c). (10 marks) Suppose that a sample of n independent values is generated from $(\Omega, \mathcal{A}, P)$ given in 1(a) and the values Y_i = the number of values in $((i-1)/4, i/4)$ are recorded for $i = 1, 2, 3, 4$. Determine the probability distribution of $Y = (Y_1, Y_2, Y_3, Y_4)$.

The probability that $\omega \in (\frac{i-1}{4}, \frac{i}{4}]$ equals $\int_{\frac{i-1}{4}}^{\frac{i}{4}} d\omega = \frac{i}{4} - \frac{(i-1)}{4} = \frac{1}{4}$ for $i = 1, 2, 3, 4$.

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Then $(Y_1, Y_2, Y_3, Y_4) \sim$ multinomial $(n, 1/4, 1/4, 1/4, 1/4)$ with probability function

$$p_{(Y_1,Y_2,Y_3,Y_4)}(y_1, y_2, y_3, y_4) = \binom{n}{y_1, y_2\ y_3\ y_4} \left(\tfrac{1}{4}\right)^{y_1} \left(\tfrac{1}{4}\right)^{y_2} \left(\tfrac{1}{4}\right)^{y_3} \left(\tfrac{1}{4}\right)^{y_4}.$$

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$$= \binom{n}{y_1\ y_2\ y_3\ y_4} \left(\tfrac{1}{4}\right)^{y_1+y_2+y_3+y_4}$$

$$= \binom{n}{y_1\ y_2\ y_3\ y_4} \left(\tfrac{1}{4}\right)^{n}.$$

1(d). (10 marks) For Y in 1(c), determine the conditional distribution of $Y \mid Z = z$ where $Z = Y_1 + Y_2$.

The random variable Z just counts the number of occurences of ω's in $(0,\frac{1}{4}] \cup (\frac{1}{4},\frac{1}{2}] = (0,\frac{1}{2}]$ and $P((0,\frac{1}{2}]) = \frac{1}{2}$. Therefore,

$Z \sim \text{binomial}(n, \frac{1}{2}) = \text{multinomial}(n, \frac{1}{2}, \frac{1}{2})$.

5 This implies that $Y|Z$ has probability function

$$p_{Y|Z=z}(y_1, y_2, y_3, y_4) = \frac{\binom{n}{y_1\, y_2\, y_3\, y_4}(\frac{1}{4})^n}{\binom{n}{z}(\frac{1}{2})^n}$$

5 $$= \left(\frac{n!}{y_1!\,y_2!\,y_3!\,y_4!}\right)\left(\frac{z!\,(n-z)!}{n!}\right)$$

$$= \binom{z}{y_1\, y_2}\left(\tfrac{1}{2}\right)^z \binom{n-z}{y_3\, y_4}\left(\tfrac{1}{2}\right)^{n-z}$$ for $y_1, y_2, y_3, y_4 \in \{0, 1, \ldots, n\}$, $y_1 + y_2 = z$, $y_3 + y_4 = n - z$

$\therefore Y|Z=z$ is s.t. $(Y_1, Y_2) \sim \text{binomial}(z, \frac{1}{2})$ stat. ind. of $(Y_3, Y_4) \sim \text{binomial}(n-z, \frac{1}{2})$.

2(a). (10 marks) Suppose that

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2(\mu, \Sigma) \text{ where } \mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}.$$

Determine the probability distribution of

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{X}.$$

Putting $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ from results in

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class we have $Y = AX \sim N_2(A\mu, A\Sigma A')$

where $A\mu = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and

$$A\Sigma A' = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

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$$= \frac{1}{2}\begin{pmatrix} 4 & -4 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 8 & 0 \\ 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

2(b). (10 marks) Are Y_1 and Y_2 in 2(a) statistically independent? Justify your answer.

Yes Y_1 and Y_2 are statistically independent since from results in class we have

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$Y_1 \sim N(\frac{-1}{\sqrt{2}}, 4)$, $Y_2 \sim N(\frac{1}{\sqrt{2}}, 2)$ and the joint density of (Y_1, Y_2) is

$$(2\pi)^{-1}\left(\det\begin{pmatrix}4 & 0\\ 0 & 2\end{pmatrix}\right)^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}\begin{pmatrix}y_1 - 1/\sqrt{2}\\ y_2 - 1/\sqrt{2}\end{pmatrix}'\begin{pmatrix}4 & 0\\ 0 & 2\end{pmatrix}^{-1}\begin{pmatrix}y_1 - \frac{1}{\sqrt{2}}\\ y_2 - \frac{1}{\sqrt{2}}\end{pmatrix}\right\}$$

$$= (2\pi)^{-1}\frac{1}{\sqrt{8}}\exp\left\{-\frac{1}{2}\frac{(y_1 - 1/\sqrt{2})^2}{4}\right\}\exp\left\{-\frac{1}{2}\frac{(y_2 - 1/\sqrt{2})^2}{2}\right\}$$

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$$= (2\pi)^{-\frac{1}{2}}\frac{1}{\sqrt{4}}\exp\left\{-\frac{1}{2}\frac{(y_1 - 1/\sqrt{2})^2}{4}\right\}(2\pi)^{-\frac{1}{2}}\frac{1}{\sqrt{2}}\exp\left\{-\frac{1}{2}\frac{(y_2 - 1/\sqrt{2})^2}{2}\right\}$$

$$= f_{Y_1}(y_1)\, f_{Y_2}(y_2)$$

2(c). (10 marks) Determine $\Sigma^{1/2}$ for Σ specified in 2(a).

Since $A \Sigma A' = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$ and A is orthogonal

6 matrix, we have that $\Sigma = A' \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} A$ is the spectral decomposition of Σ. Therefore,

5 $$\Sigma^{1/2} = A' \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}^{1/2} A = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2 & \sqrt{2} \\ -2 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2+\sqrt{2}}{2} & \frac{\sqrt{2}-2}{2} \\ \frac{\sqrt{2}-2}{2} & \frac{2+\sqrt{2}}{2} \end{pmatrix}$$

2(d). (10 marks) For $\mathbf{X}$ as specified in 2(a) determine the conditional distribution of X_1 given $Z = X_1 + X_2 = z$.

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \right).$$

Therefore (by results established in class)

$$\begin{pmatrix} X_1 \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

5 $$\sim N_2\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)$$

$$= N_2\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix} \right)$$

Now putting $\underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$

with $\Sigma_{11} = 3$, $\Sigma_{12} = 2$, $\Sigma_{22} = 4$ then

5 $$X_1 \mid Z = z \sim N\left(\mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (z - \mu_2), \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}' \right)$$

$$= N\left(1 + \tfrac{2}{4}(z-1), 3 - 2\left(\tfrac{1}{4}\right) 2 \right)$$

$$= N\left(\tfrac{1}{2} + \tfrac{1}{2} z, 2 \right).$$

3(a). (10 marks) Suppose that $T = \{a, b, c\}$ and we define

$$X_a \sim N(0,1), X_b \sim N(0,2), X_c \sim N(0,3)$$
$$\begin{pmatrix} X_a \\ X_b \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\right),$$
$$\begin{pmatrix} X_a \\ X_c \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\right),$$
$$\begin{pmatrix} X_b \\ X_c \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\right),$$
$$\begin{pmatrix} X_a \\ X_b \\ X_c \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}\right).$$

Does this define a stochastic process $\{(t, X_t) : t \in T\}$? Justify your conclusion.

5 This does not define a valid stochastic process by the Kolmogorov Consistency Theorem since

$$\begin{pmatrix} X_a \\ X_b \\ X_c \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}\right)$$

5 implies

$$\begin{pmatrix} X_a \\ X_c \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\right) \neq N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\right).$$

3(b). (10 marks) Suppose that $(\Omega, \mathcal{A}, P) = (\{-1, 0, 1\}, 2^{\Omega}, P)$ where P is the uniform probability measure. Then for $T = \{1, 2\}$ define

$$X_1(\omega) = \begin{cases} 0 & \text{if } \omega = -1 \\ 0 & \text{if } \omega = 0 \\ 2 & \text{if } \omega = 1 \end{cases} \quad \text{and } X_2(\omega) = \begin{cases} 2 & \text{if } \omega = -1 \\ 0 & \text{if } \omega = 0 \\ 0 & \text{if } \omega = 1 \end{cases}.$$

Determine the marginal and joint marginal distributions of (X_1, X_2) and plot the sample function when $\omega = 0$.

$P(X_1 = 0) = \frac{2}{3}$, $P(X_1 = 2) = \frac{1}{3}$, $P(X_1 = x) = 0$ otherwise

$P(X_2 = 0) = \frac{2}{3}$, $P(X_2 = 2) = \frac{1}{3}$, $P(X_2 = x) = 0$ otherwise

5 $P\left(\binom{X_1}{X_2} = \binom{0}{0}\right) = \frac{1}{3}$, $P\left(\binom{X_1}{X_2} = \binom{0}{2}\right) = \frac{1}{3}$

$P\left(\binom{X_1}{X_2} = \binom{2}{0}\right) = \frac{1}{3}$ and $P\left(\binom{X_1}{X_2} = \binom{x_1}{x_2}\right) = 0$ otherwise

For $\omega = 0$, $X_1(\omega) = 0$, $X_2(\omega) = 0$ so the sample function is plotted as

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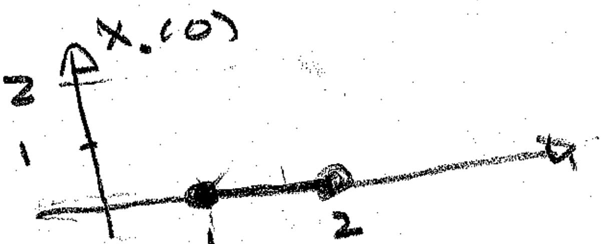

Note: joining the points (1,0) and (2,2) by a line (as in the graph) is not necessary but is usually done.