UNIVERSITY OF TORONTO SCARBOROUGH

Department of Computer and Mathematical Sciences

STAC62: Probability and Stochastic Processes I

Midterm Test 1 2023

Duration: 1 hour

Solutions

1(a). (10 marks) Suppose that $\Omega = \{1,2,3,4,5,6\}$. Prove or disprove that

$$\mathcal{A} = \{\phi, \{1,2\}, \{3,4\}, \{5\}, \{6\}, \Omega\}$$

is a σ-algebra on Ω.

This is not a σ-algebra because it is not closed under complementation.

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5 For example $\{6\}^c = \{1,2,3,4,5\} \notin \mathcal{A}$.

1(b). (10 marks) Suppose that $\mathcal{C} = \{\{1\}, \{2\}\}$. What is the smallest σ-algebra on Ω in question 1(a) that contains $\mathcal{C}$?

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$$A(\mathcal{C}) = \{\phi, \{1\}, \{2\}, \{1,2\}, \{3,4,5,6\}, \{1,3,4,5,6\}, \{2,3,4,5,6\}, \Omega\}$$

2. (10 marks) Consider the probability model $(\Omega \times \Omega, \mathcal{A} \times \mathcal{A}, P)$ where $\mathcal{A}$ is a σ-algebra on Ω and $\mathcal{A} \times \mathcal{A}$ is the smallest σ-algebra on $\Omega \times \Omega$ containing all sets of the form $A \times B$ for $A, B \in \mathcal{A}$. For $A \times B \in \mathcal{A} \times \mathcal{A}$ prove that $(A \times B)^c = (A^c \times B) \cup (A \times B^c) \cup (A^c \times B^c)$ and

$$P((A \times B)^c) = P((A^c \times B)) + P((A \times B^c)) + P((A^c \times B^c)).$$

If $(\omega_1, \omega_2) \in (A \times B)^c$ then $(\omega_1, \omega_2) \notin A \times B$ which implies $\omega_1 \notin A$ or $\omega_2 \notin B$ which implies
5 $(\omega_1, \omega_2) \in (A^c \times B) \cup (A \times B^c) \cup (A^c \times B^c)$. If $(\omega_1, \omega_2) \in (A^c \times B) \cup (A \times B^c) \cup (A^c \times B^c)$ then (ω_1, ω_2)
10 is in at least one of these sets which implies
6 $\omega_1 \in A^c$ or $\omega_2 \in B^c$ which implies $(\omega_1, \omega_2) \notin A \times B$ or $(\omega_1, \omega_2) \in (A \times B)^c$. Therefore,
$(A \times B)^c = (A^c \times B) \cup (A \times B^c) \cup (A^c \times B^c)$.

Now note that sets $A^c \times B$, $A \times B^c$, $A^c \times B^c$ are disjoint sets so by the additivity
10 of P we have that

$$P((A \times B)^c) = P(A^c \times B) + P(A \times B^c) + P(A^c \times B^c).$$

3. (10 marks) For subsets A_n of a set Ω prove that $\{\liminf_{n\to\infty} A_n\}^c = \limsup_{n\to\infty} A_n^c$. (Hint: Use properties of complementation as it acts on union and intersection of sets.)

$$\{\liminf_{n\to\infty} A_n\}^c = \{\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i\}^c$$

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$$= \bigcap_{n=1}^{\infty} \left(\bigcap_{i=n}^{\infty} A_i\right)^c \quad \text{since } \left(\bigcup_{i=1}^{\infty} B_i\right)^c = \bigcap_{i=1}^{\infty} B_i^c$$

$$= \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i \quad \text{since } \left(\bigcap_{i=1}^{\infty} B_i\right)^c = \bigcup_{i=1}^{\infty} B_i^c$$

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$$= \limsup_{n\to\infty} A_n$$

4. Suppose that $(\Omega, \mathcal{A}, P)$ is a probability model and $A_1, A_2 \in \mathcal{A}$. Define the *symmetric difference* of these sets by $A_1 \Delta A_2 = (A_1 \cap A_2^c) \cup (A_1^c \cap A_2)$.

4 (a). (5 marks) Draw a Venn diagram and indicate the set $A_1 \Delta A_2$ on it.

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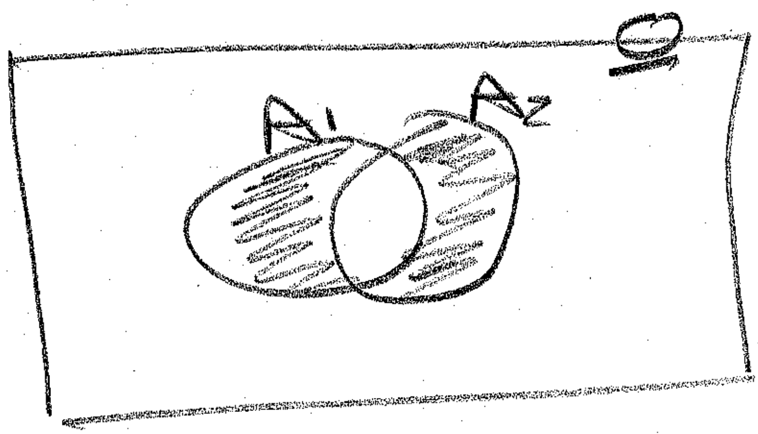

$A_1 \Delta A_2$ ≡

4(b). (5 marks) Prove $A_1 \Delta A_2 \in \mathcal{A}$.

Since $A_1, A_2 \in \mathcal{A}$ we have $A_1^c, A_2^c \in \mathcal{A}$

3 and so $A_1 \cap A_2^c, A_1^c \cap A_2 \in \mathcal{A}$ which

implies $A_1 \Delta A_2 = (A_1^c \cap A_2) \cup (A_1 \cap A_2^c)$

since $\mathcal{A}$ is closed under complementation

2 and finite unions and intersections.

4(c) (5 marks) What are $A_1 \Delta \phi$ and $A_1 \Delta \Omega$?

$A_1 \Delta \phi = (A_1^c \cap \phi) \cup A_1 \cap \phi^c$

$= \phi \cup A_1$ since $\phi^c = \Omega$

$= A_1$

3

$A_1 \Delta \Omega = (A_1^c \cap \Omega) \cup (A_1 \cap \Omega^c)$

$= A_1^c \cup \phi$, since $\Omega^c = \phi$

$= A_1^c$

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4(d). (5 marks) Prove $P(A_1 \Delta A_2) = P(A_1 \cap A_2^c) + P(A_1^c \cap A_2)$.

Since $A_1 \Delta A_2 = (A_1^c \wedge A_2) \cup (A_1 \wedge A_2^c)$

3 and $(A_1^c \wedge A_2) \wedge (A_1 \wedge A_2^c) = A_1 \wedge A_1^c \wedge A_2 \wedge A_2^c = \phi$

the additivity of P implies

2 $P(A_1 \Delta A_2) = P(A_1^c \wedge A_2) + P(A_1 \wedge A_2^c)$

5. Suppose that $(\Omega, \mathcal{A}, P)$ is a probability model and $A, B \in \mathcal{A}$ with $0 < P(B) < 1$.

5(a). (~~15~~ 20 marks) Prove that A and B are statistically independent iff $P(A \mid B) = P(A \mid B^c)$.

By an Exercise the events A and B are statistically independent iff each element of $\{\phi, A, A^c, \Omega\}$ is statistically independent of each element of $\{\phi, B, B^c, \Omega\}$.

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This implies by a result in the notes that

$$P(A|B) = P(A) \quad \text{and} \quad P(A|B^c) = P(A)$$

and so $P(A|B) = P(A|B^c)$.

Now suppose $P(A|B) = P(A|B^c)$.

Then $P(A|B) = \dfrac{P(A \wedge B)}{P(B)} = \dfrac{P(A \wedge B^c)}{P(B^c)} = P(A|B^c)$.

Now $P(B^c) = 1 - P(B)$ and $P(A \wedge B^c) = P(A) - P(A \wedge B)$

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and so $\quad P(A \wedge B)(1 - P(B)) = (P(A) - P(A \wedge B))\,P(B)$

or $\quad P(A \wedge B) = P(A)P(B)$ so

A and B are statistically independent.

5(b) (10 marks) Explain what the result in 5(a) means in terms of probability measuring degrees of belief in an application.

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The statistical independence of events A and B occurs iff our belief in the truth of A does not depend on whether B is true or false.