Probability and Stochastic Processes I - Lecture 4

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- recall

Definition 1.6.1 When $(\Omega, \mathcal{A}, P)$ is a probability model and $C \in \mathcal{A}$ satisfies $P(C) > 0$, then the *conditional probability model given* C is $(\Omega, \mathcal{A}, P(\cdot \mid C))$ where $P(\cdot \mid C) : \mathcal{A} \to [0, 1]$ is given by

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)}. \blacksquare$$

- this leads to the concept of statistical independence (no change in belief, no relationship, no evidence, ...)

- basic idea follows from conditional probability as A and C are statistically independent whenever $P(A \mid C) = P(A)$ so knowing that C is true does not change our belief that A is true

- note $P(A \mid C) = P(A)$ implies

$$P(A \cap C) = P(A)P(C)$$

- but to cover the case when $P(C) = 0$ the following definition is used

Definition 1.7.1 When $(\Omega, \mathcal{A}, P)$ is a probability model and $A, C \in \mathcal{A}$, then A and C are *statistically independent* whenever $P(A \cap C) = P(A)P(C)$. ■

- it is immediate from the definition that whenever $P(C) > 0$ then

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{P(A)P(C)}{P(C)} = P(A)$$

- if $P(C) = 0$, then $P(A \cap C) \leq P(C)$, because $A \cap C \subset C$, and thus $P(A \cap C) = P(A)P(C) = 0$ so A and C are stat. ind.

Exercise 1.7.1 If A and B are stat. ind. then show that every element of $\{\phi, A, A^c, \Omega\}$ (the σ-algebra generated by A) is stat. ind. of every element of $\{\phi, B, B^c, \Omega\}$ (the σ-algebra generated by B). So we say the two σ-algebras are stat. ind.

- now consider the stat. independence of more than two events

- it turns out to be easier to define what it means for an arbitrary collection of σ-algebras to be mutually statistically independent

Definition 1.7.2 When $(\Omega, \mathcal{A}, P)$ is a probability model and $\{\mathcal{A}_\lambda : \lambda \in \Lambda\}$ is a collection of sub σ-algebras of $\mathcal{A}$, then the $\mathcal{A}_\lambda$ are *mutually statistically independent* whenever for any n and distinct $\lambda_1, \ldots, \lambda_n \in \Lambda$ and any $A_1 \in \mathcal{A}_{\lambda_1}, \ldots, A_n \in \mathcal{A}_{\lambda_n}$, then

$$P(A_1 \cap \cdots \cap A_n) = \prod_{i=1}^{n} P(A_i). \blacksquare$$

- **note** - this tells us how to define the mut. stat. ind. of events $A_1, \ldots, A_n \in \mathcal{A}$, namely, the σ-algebras $\{\phi, A_i, A_i^c, \Omega\}$ for $i = 1, \ldots, n$ must be mut. stat. ind.

Example 1.7.1 $P(A \cap B \cap C) = P(A)P(B)P(C)$ *does not imply mutual independence*

- consider tossing a fair coin 3 times where head $= 1$ and tail $= 0$ so

$$\Omega = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$$

with$\mathcal{A} = 2^{\Omega}$ and put

$$\begin{aligned} A &= \text{"first toss is a head"} \\ &= \{(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}, \\ B &= \text{"last two tosses are tails or first two tosses are heads"} \\ &= \{(0,0,0),(1,0,0),(1,1,0),(1,1,1)\}, \\ C &= \text{"last two tosses are different"} \\ &= \{(0,0,1),(0,1,0),(1,0,1),(1,1,0)\}, \end{aligned}$$

- with uniform P then $P(A) = P(B) = P(C) = 1/2$,

$$P(A \cap B \cap C) = P(\{(1,1,0)\}) = 1/8 = P(A)P(B)P(C)$$

but

$$\begin{aligned} P(A \cap B \cap \Omega) &= P(\{(1,0,0),(1,1,0),(1,1,1)\}) \\ &= 3/8 \neq P(A)P(B)P(\Omega) = 1/4 \end{aligned}$$

and so A, B and C are not mut. stat. ind. ■

Example 1.7.2 *Pairwise independence does not imply mutual independence.*

- suppose $\Omega = \{1,2,3,4\}, \mathcal{A} = 2^{\Omega}, A = \{1,2\}, B = \{1,3\}, C = \{1,4\}$

- $\Lambda = \{a,b,c\}, \mathcal{A}_a = \mathcal{A}(\{A\}) = \{\phi, \{1,2\}, \{3,4\}, \Omega\}$ and similarly $\mathcal{A}_b = \mathcal{A}(\{B\}), \mathcal{A}_c = \mathcal{A}(\{C\})$

- assign $P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\} = 1/4$ (the uniform) so

$$\begin{aligned} P(A) &= P(B) = P(C) = 1/2 \\ P(\{1\}) &= P(A \cap B) = P(A \cap C) = P(B \cap C) = 1/4 \end{aligned}$$

- so A and B are stat. ind., and similarly A and C are stat. ind. and B and C are stat. ind.

- this implies $\mathcal{A}(\{A\}), \mathcal{A}(\{B\})$ and $\mathcal{A}(\{C\})$ are pairwise independent but

$$P(\{1\}) = P(A \cap B \cap C) \neq 1/8 = P(A)P(B)P(C)$$

and so $\mathcal{A}(\{A\}), \mathcal{A}(\{B\})$ and $\mathcal{A}(\{C\})$ are not mutually statistically independent ■

Exercise 1.7.2 Suppose $\Omega = \{1,2\} \times \{1,2\}, \mathcal{A} = 2^\Omega$ and P is the uniform probability measure.
(a) Show that $\mathcal{A}_1 = \{\phi, \{1\} \times \{1,2\}, \{2\} \times \{1,2\}, \Omega\}$ and $\mathcal{A}_2 = \{\phi, \{1,2\} \times \{1\}, \{1,2\} \times \{2\}, \Omega\}$ are sub σ-algebras of $\mathcal{A}$.
(b) Determine whether or not $\mathcal{A}_1$ and $\mathcal{A}_2$ are statistically independent.

8. Exercises for Chapter 1

Exercise 1.8.1 Evans and Rosenthal (E&R)1.3.8

- the following 3 exercises deal with the *inclusion-exclusion formulas*

Exercise 1.8.2 Prove:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Exercise 1.8.3 Generalize the result in Ex. 1.8.2 to $A_1, \ldots, A_n$ and prove using induction.

Exercise 1.8.4 Note that $P(A \cap B) = P(A) + P(B) - P(A \cup B)$. Generalize this to three events A, B, C. State the general result for $A_1, \ldots, A_n$.

Exercise 1.8.5 Suppose $A_n = (-1/n, 1 + (n-1)/n]$. Determine $\liminf A_n$ and $\limsup A_n$. If this sequence of Borel sets converges then determine the limiting probability when P is the $N(0,1)$ probability measure. Justify all your results.

Exercise 1.8.6 (E&R) 1.6.10

Exercise 1.8.7 (E&R) 1.6.11 Hint: for events $A_1, A_2, \ldots$ what does the sequence of events $B_n = \cup_{i=1}^{n} A_i$ converge to?