Probability and Stochastic Processes I - Lecture 25

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Example V.2.1 *Wiener process (Brownian motion)*

Definition V.2.1 A s.p. $\{(t, W_t) : t \geq 0\}$ is a *standard Wiener process* if (i) $P(W_0 = 0) = 1$ (ii) the process has *independent increments*, namely, for any $0 < t_1 < \cdots < t_k$ then $W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_k} - W_{t_{k-1}}$ are mutually stat. ind. and (iii) $W_t - W_s \sim N(0, t-s)$ for any $0 \leq s \leq t$. ∎

- then $\{(t, X_t) : t \geq 0\}$ with $X_t = \tau W_t \sim N(0, \tau^2(t-s))$ is a general Wiener process

Proposition V.2.1 $\{(t, X_t) : t \geq 0\}$ is a Gaussian process with mean function 0 and autocovariance function $\sigma(s,t) = \tau^2 \min(s,t)$ (and so is not stationary).

Proof: For any $0 < t_1 < \cdots < t_n$ and $c_1, \ldots, c_n \in R^1$

$$\sum_{i=1}^n c_i X_{t_i} = \tau \sum_{i=1}^n c_i W_{t_i} = \tau[c_n(W_{t_n} - W_{t_{n-1}}) + (c_{n-1} + c_n)(W_{t_{n-1}} - W_{t_{n-2}}) + \cdots + (c_1 + \cdots + c_n)W_{t_1}$$

$$\sim N\left(0, \tau^2 \sum_{i=1}^{n} \left(\sum_{j=1}^{n-i+1} c_j\right)^2 (t_i - t_{i-1})\right)$$

and so $(X_{t_1}, \ldots, X_{t_n})'$ is multivariate normal since every linear combination is normal (Prop. III.9.8). Also,

$$\begin{aligned} & \sigma(s,t) = E(X_s X_t) = \tau^2 E(W_s W_t) \overset{s \leq t}{=} \tau^2 E(W_s(W_s + W_t - W_s)) \\ = & \ \tau^2 E(W_s^2) + \tau^2 E(W_s(W_t - W_s)) = \tau^2 s + \tau^2 0 = \tau^2 s = \tau^2 \min(s,t). \end{aligned}$$

Therefore, $(X_{t_1}, \ldots, X_{t_n})' \sim N_n(\mathbf{0}, \tau^2(\min(t_i, t_j)))$ and so by KCT this is a Gaussian process. ■

Proposition V.2.2 There exists a version of $\{(t, W_t) : t \geq 0\}$ also satisfying (iv) $P(W_t$ is continuous in $t) = 1$ and (v) $P(W_t$ is nowhere differentiable in $t) = 1$.
Proof: Accept.

- how does Brownian motion arise? as a limiting process

- suppose $Z_1, Z_2, \ldots$ $i.i.d.$ with mean 0 and variance 1 put $S_0 = 0$ and $S_n = \sum_{i=1}^{n} Z_i$ a random walk (e.g., a simple symmetric random walk when $Z_i \sim -1 + 2\text{Bernoulli}(1/2)$)

Proposition V.2.3 (*Donsker's Theorem or Invariance Principle*)

$$\left\{\left(t, n^{-1/2}S_{\lfloor nt\rfloor}\right) : t \in [0,1]\right\} \xrightarrow{d} \{(t, W_t) : t \in [0,1]\}$$

- space is shrunk by factor $1/\sqrt{n}$ and time speeded up by factor n

- for $T = [0, T_0]$ put $\Delta T_0 = T_0/n$, then

$$\begin{aligned} & \left\{\left(t, (\Delta T_0)^{1/2}S_{\lfloor t/\Delta T_0\rfloor}\right) : t \in [0, T_0]\right\} \\ = & \left\{\left(t, {T_0}^{1/2}n^{-1/2}S_{\lfloor nt/T_0\rfloor}\right) : t/T_0 \in [0,1]\right\} \xrightarrow{d} \{(t, W_t) : t \in [0, T_0]\} \end{aligned}$$

- sample paths $t \to n^{-1/2}S_{\lfloor nt\rfloor}$ are not continuous but

$$t \to n^{-1/2}[(1 - nt + \lfloor nt\rfloor)S_{\lfloor nt\rfloor} + (nt - \lfloor nt\rfloor)S_{\lfloor nt\rfloor+1}]$$

has continuous sample paths and the same convergence result applies

- these results also tell us how to simulate (approximately) from $\{(t, W_t) : t \geq 0\}$

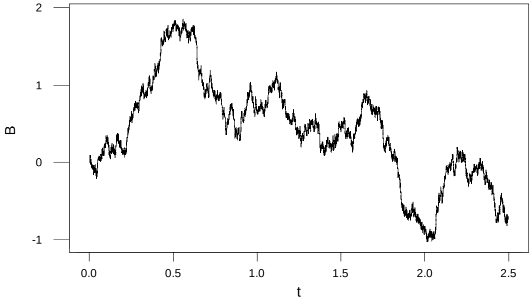

Figure: Simulated Brownian motion.

- define a *diffusion process* as $\{(t, X_t) : t \geq 0\}$ by $X_t = \alpha + \delta t + \sigma W_t$ (stock market) where α = initial value, δ = drift and σ = volatility ■

Exercise V.2.1 E&R 11.5.7, 11.5.8, 11.5.12, 11.5.13.