Probability and Stochastic Processes I
Lecture 2

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I.4 Borel Sets

Proposition 1.4.1 If $\{\mathcal{A}_\lambda : \lambda \in \Lambda\}$ is a family of σ-algebras on Ω, then $\cap_{\lambda\in\Lambda}\mathcal{A}_\lambda$ is a σ-algebra on Ω.
Proof: We check that $\cap_{\lambda\in\Lambda}\mathcal{A}_\lambda$ has all the necessary properties to be a σ-algebra.
(i) Since $\phi \in \mathcal{A}_\lambda$ for every λ it follows that $\phi \in \cap_{\lambda\in\Lambda}\mathcal{A}_\lambda$.
(ii) Suppose $A_1, A_2, \ldots \in \mathcal{A}_\lambda$ for every λ. Then $\cup_{i=1}^{\infty} A_i \in \mathcal{A}_\lambda$ for every λ since $\mathcal{A}_\lambda$ is a σ-algebra. This in turn implies that $\cup_{i=1}^{\infty} A_i \in \cap_{\lambda\in\Lambda}\mathcal{A}_\lambda$.
(iii) Suppose $A \in \mathcal{A}_\lambda$ for every λ and so $A^c \in \mathcal{A}_\lambda$ for every λ which implies $A^c \cap_{\lambda\in\Lambda} \mathcal{A}_\lambda$.
Therefore $\cap_{\lambda\in\Lambda}\mathcal{A}_\lambda$ is a σ-algebra on Ω. ■

Example 1.4.1 $\Omega = \{1, 2, 3, 4\}$

- $\mathcal{A}_1 = \{\phi, \{1, 2\}, \{3, 4\}, \Omega\}, \mathcal{A}_2 = \{\phi, \{1\}, \{2, 3, 4\}, \Omega\}$

- $\mathcal{A}_1 \cap \mathcal{A}_2 = \{\phi, \Omega\}$■

- **note** - Λ does not have to be countable (can be placed in 1-1 correspondence with the natural numbers)

Definition 1.4.1 For any $\mathcal{C} \subseteq 2^{\Omega}$ ($\mathcal{C}$ is a set consisting of subsets of Ω) the *σ-algebra generated by $\mathcal{C}$*, denoted $\mathcal{A}(\mathcal{C})$, is the intersection of all σ-algebras containing $\mathcal{C}$. ■

- clearly $\mathcal{A}(\mathcal{C})$ is the smallest σ-algebra on Ω that contains all the subsets in $\mathcal{C}$ (any σ-algebra containing $\mathcal{C}$ is in the intersection)

Example 1.4.2 $\Omega = \{1, 2, 3, 4\}$

- if $\mathcal{C} = \{\{1, 2\}\}$, then $\mathcal{A}(\mathcal{C}) = \{\phi, \{1, 2\}, \{3, 4\}, \Omega\}$

- if $\mathcal{C} = \{\{1\}, \{2\}\}$, then

$$\mathcal{A}(\mathcal{C}) = \{\phi, \{1\}, \{2\}, \{1, 2\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \Omega\}$$

■

Exercise 1.4.1 If $\Omega = \{1, 2, 3, 4, 5\}$ and $\mathcal{C} = \{\{1, 2\}, \{3, 4\}, \{3, 4, 5\}\}$ what is $\mathcal{A}(\mathcal{C})$?

- the Borel sets $\mathcal{B}^k$ is the most commonly used σ-algebra when $\Omega = R^k$

Definition 1.4.2 The *Borel sets* $\mathcal{B}^k$ is the smallest σ-algebra on R^k that contains all rectangles of the form

$$\begin{aligned}(\mathbf{a}, \mathbf{b}] &= X_{i=1}^k (a_i, b_i] = (a_1, b_1] \times \cdots \times (a_k, b_k] \\ &= \{(x_1, \ldots, x_k) : a_i < x_i \leq b_i\}\end{aligned}$$

where $\mathbf{a} = (a_1, \ldots, a_k)', \mathbf{b} = (b_1, \ldots, b_k)' \in R^k$. ■

- note - elements of R^k will be written as columns and $'$ denotes transpose

- since 2^{R^k} contains all such rectangles this proves $\mathcal{B}^k \neq \phi$

- **fact:** $\mathcal{B}^k \neq 2^{R^k}$, namely, there is a subset $A \subseteq R^k$ and A is not a Borel set

Exercise 1.4.3 A rectangle in R^1 is just an interval such as $(a, b]$. Prove that $\{a\} \in \mathcal{B}^1$ for all $a \in R^1$ (consider the intersection of intervals $(a - 1/n, a])$. Prove that $[a, b], (a, b), [a, b), (-\infty, b], (a, \infty) \in \mathcal{B}^1$ for all $a, b \in R^1$.

Exercise 1.4.4 For $\mathbf{a}, \mathbf{b} \in R^2$ prove that $\{\mathbf{a}\}, (\mathbf{a}, \mathbf{b}), [\mathbf{a}, \mathbf{b}), [\mathbf{a}, \mathbf{b}], (\infty, \mathbf{b}] \in \mathcal{B}^2$ and also $(a_1, b_1] \times [a_2, b_2) \in \mathcal{B}^2$.

- loosely speaking any set you can define explicitly is a Borel set

- for example, a *ball* of radius r and centered at $\mathbf{x}_0$, namely,

$$B_r(\mathbf{x}_0) = \{\mathbf{x} : (\mathbf{x} - \mathbf{x}_0)'(\mathbf{x} - \mathbf{x}_0) = \sum_{i=1}^{k} (x_i - x_{0i})^2 \leq r^2\} \in \mathcal{B}^k$$

and a *sphere* of radius r and centered at $\mathbf{x}_0$, namely,

$$S_r(\mathbf{x}_0) = \{\mathbf{x} : (\mathbf{x} - \mathbf{x}_0)'(\mathbf{x} - \mathbf{x}_0) = r^2\} \in \mathcal{B}^k$$

- also, (nice) transformations of Borel sets are typically Borel sets

- for example, let

$$A = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kk} \end{pmatrix} = (a_{ij}) \in R^{k \times k}$$

be an invertible $k \times k$ matrix and $\mathbf{b} \in R^k$

- define a new set

$$\begin{aligned} AB_r(\mathbf{x}_0)+\mathbf{b} &= \{\mathbf{y} : \mathbf{y} = A\mathbf{x}+\mathbf{b} \text{ for some } \mathbf{x} \in B_r(\mathbf{x}_0)\} \\ &= \{\mathbf{y} : (A^{-1}(\mathbf{y}-\mathbf{b})-\mathbf{x}_0)'(A^{-1}(\mathbf{y}-\mathbf{b})-\mathbf{x}_0) \le r^2\} \\ &= \{\mathbf{y} : (\mathbf{y}-\mathbf{b}-A\mathbf{x}_0)'(A^{-1})'A^{-1}(\mathbf{y}-\mathbf{b}-A\mathbf{x}_0) \le r^2\} \\ &= \{\mathbf{y} : (\mathbf{y}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{y}-\boldsymbol{\mu}) \le r^2\} \\ &= E_r(\boldsymbol{\mu},\Sigma) \in \mathcal{B}^k \end{aligned}$$

where $\boldsymbol{\mu} = A\mathbf{x}_0 + \mathbf{b}$ and $\Sigma = ((A^{-1})'A^{-1})^{-1} = AA'$

- $E_r(\boldsymbol{\mu},\Sigma)$ is the *ellipsoidal region* with center at $\boldsymbol{\mu}$ and axes and orientation determined by Σ and r

- note that

$\Sigma \in R^{k\times k}$ is *symmetric* since $\Sigma' = (AA')' = (A')'A' = AA' = \Sigma$,

Σ is *invertible* since $((A^{-1})'A^{-1})\Sigma = ((A^{-1})'A^{-1})AA' = I$ so $\Sigma^{-1} = (A^{-1})'A^{-1}$

Σ is *positive definite* since for any $\mathbf{w} \in R^k$ then

$$\mathbf{w}'\Sigma\mathbf{w} = \mathbf{w}'AA'\mathbf{w} = ||A'\mathbf{w}||^2 \geq 0$$

and is 0 only when $\mathbf{w} = \mathbf{0}$ since A is invertible which implies A' is invertible

Exercise 1.4.5 Suppose

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and $\mathbf{x}_0 = (0,0)', \mathbf{b} = (1,1)'$. Write out $E_{3/2}(\boldsymbol{\mu},\Sigma)$ in terms of an inequality that the coordinates y_1 and y_2 have to satisfy.

Example 1.4.3 Uniform probability measure on $[0,1]^k$

- suppose $\Omega = [0,1]^k = [\mathbf{0},\mathbf{1}]$ where $\mathbf{0} = (0,\ldots,0)', \mathbf{1} = (1,\ldots,1)' \in R^k$

- for $[\mathbf{a},\mathbf{b}] \subseteq [\mathbf{0},\mathbf{1}]$ define $P([\mathbf{a},\mathbf{b}]) = \prod_{i=1}^{k}(b_i - a_i) =$ the volume of the k-cell

- $P([\mathbf{0},\mathbf{1}]) = \prod_{i=1}^{k}(1-0) = 1$ and volume is additive

- fact: there is a unique probability measure P on the Borel subsets of $[\mathbf{0},\mathbf{1}]$ that agrees with volume measure on the k-cells

- the Borel subsets of $[\mathbf{0},\mathbf{1}]$ are given by $\mathcal{B}^k \cap [\mathbf{0},\mathbf{1}]$

- this P is called the uniform probability measure on $[\mathbf{0},\mathbf{1}]$ and $([\mathbf{0},\mathbf{1}], \mathcal{B}^k \cap [\mathbf{0},\mathbf{1}], P)$ is the uniform probability model on $[\mathbf{0},\mathbf{1}]$

- note that $P(\{\mathbf{a}\}]) = P([\mathbf{a},\mathbf{a}]) = 0$ for every $\mathbf{a} \in [\mathbf{0},\mathbf{1}]$

- this is an example of a continuous probability measure but recall this is an approximation to a discrete probability measure on a large (finite) number of equispaced points in $[\mathbf{0}, \mathbf{1}]$ ■

Exercise 1.4.6 Define a uniform probability measure on $[\mathbf{a}, \mathbf{b}] \subseteq R^k$ when $a_i \leq b_i$ for $i = 1, \ldots, k$.