Probability and Stochastic Processes I - Lecture 14

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- we want to prove that E is linear and note $E(a) = a$ for any constant a since a constant is a simple function

Lemma III.1.2 If Y, Z are nonnegative r.v.'s and (i) $a, b \geq 0$, then $E(aY + bZ) = aE(Y) + bE(Z)$ and (ii) if $Y \leq Z$, then $0 \leq E(Y) \leq E(Z)$.
Proof: Choose nonnegative simple $Y_n \uparrow Y, Z_n \uparrow Z$. (i) Then $aY_n + bZ_n$ is nonnegative simple satisfying $aY_n + bZ_n \uparrow aY + bZ$ and so

$$\begin{aligned} E(aY + bZ) &= \lim_n E(aY_n + bZ_n) = a \lim_n E(Y_n) + b \lim_n E(Z_n) \\ &= aE(Y) + bE(Z). \end{aligned}$$

(ii) We have $0 \leq Y_n \leq \max\{Y_n, Z_n\}$ and $\max\{Y_n, Z_n\}$ is simple satisfying $Z_n \leq \max\{Y_n, Z_n\} \uparrow Z$. Therefore by Prop. III.1.1(ii) $0 \leq E(Y_n) \leq E(\max\{Y_n, Z_n\})$ and the result follows since $E(Y_n) \to E(Y), E(\max\{Y_n, Z_n\}) \to E(Z)$. ■

Lemma III.1.3 If Y, Z are nonnegative r.v.'s with $E(Y), E(Z)$ finite, then $E(Y - Z) = E(Y) - E(Z)$.

Proof: Put $X = Y - Z = X_+ - X_-$ so $Y + X_- = Z + X_+$ is nonnegative and $E(Y + X_-) = E(Z + X_+)$. Then by Lemma III.1.2(i)

$$\begin{aligned} E(Y + X_-) &= E(Y) + E(X_-), \\ E(Z + X_+) &= E(Z) + E(X_+). \end{aligned}$$

Also, if $X_+(\omega) > 0$, then $X_+(\omega) = Y(\omega) - Z(\omega) \leq Y(\omega) + Z(\omega)$ and so $X_+(\omega) \leq Y(\omega) + Z(\omega)$ for every ω. Using Lemma III.1.2(ii) this implies $0 \leq E(X_+) \leq E(Y) + E(Z) < \infty$ and similarly $E(X-) < \infty$. Therefore, $E(X)$ is finite and

$$\begin{aligned} E(Y - Z) &= E(X) = E(X_+) - E(X_-) \\ &= E(Z + X_+) - E(Z) - E(Y + X_-) + E(Y) \\ &= E(Y) - E(Z). \blacksquare \end{aligned}$$

Proposition III.1.4 If Y, Z are r.v.'s and $E(Y), E(Z)$ are finite, then $E(aY + bZ) = aE(Y) + bE(Z)$.
Proof: We have

$$\begin{aligned} aY + bZ &= a(Y_+ - Y_-) + b(Z_+ - Z_-) \\ &= \begin{cases} (aY_+ + bZ_+) - (aY_- + bZ_-) & \text{if } a, b \geq 0 \\ (-aY_- - bZ_-) - (-aY_+ - bZ_+) & \text{if } a < 0, b < 0 \\ (aY_+ - bZ_-) - (aY_- - bZ_+) & \text{if } a \geq 0, b < 0 \\ (-aY_- + bZ_+) - (-aY_{+-} + bZ_-) & \text{if } a < 0, b \geq 0 \end{cases} \end{aligned}$$

which is always in the form of the difference of two nonnegative r.v.'s as in Lemma III.1.3. Applying Lemmas III.1.3 and III.1.2 gives the result. ■

Example III.1.1 *St. Petersburg Paradox*

- let $\Omega = (0, \infty), \mathcal{A} = \mathcal{B}^1 \cap (0, \infty)$ and define $X : \Omega \to R^1$ by $X(\omega) = 2^{\lceil \omega \rceil}$ and $X^{-1}\{2^i\} = (i-1, i]$ and $X^{-1}\{x\} = \phi$ whenever x is not a positive integer power of 2, so X is a nonnegative r.v.

- suppose P is discrete with $P(\{i\}) = (1/2)^i$ (geometric$(1/2)$)

- putting $X_n = XI_{[0,n]}$ we see that X_n is a nonnegative simple function, $X_n \uparrow X$, and

$$E(X_n) = \sum_{i=1}^{n} 2^i \frac{1}{2^i} = n \to \infty = E(X)$$

- fair price to pay for gamble if payoff is \$$2^i$ when first head occurs on the i-th toss of a fair coin is \$$\infty$

- if we took $Y = Z = X$, then $E(Y - Z) = 0$ but $E(Y) - E(Z)$ is not defined ■

Proposition III.1.5 (i) $E(|X|) = E(X_+) + E(X_-)$ (note $E(|X|) < \infty$ implies $E(X)$ is finite)
(ii) If $X \leq Y$ with defined expectation, then $E(X) \leq E(Y)$.
(iii) If $P(X = 0) = 1$, then $E(X) = 0$.
(iv) If X and Y are equal with probability 1, namely,

$$1 = P(X = Y) = P(\{\omega : X(\omega) = Y(\omega)\}),$$

then $E(X) = E(Y)$ whenever $E(X)$ exists.
Proof: (i) This follows from Lemma III.1.2(i) since $|X| = X_+ + X_-$.

(ii) If $E(Y) = \infty$ this is obviously true and similarly if $E(X) = -\infty$. So assume neither of these cases holds which implies $E(X_-) < \infty$ and $E(Y_+) < \infty$. Now

$$X = X_+ - X_- \leq Y = Y_+ - Y_- \leq Y_+$$

and so $X_+ \leq Y_+ + X_-$ which implies $E(X_+) < \infty$ and similarly $E(Y_-) < \infty$ which implies both $E(X), E(Y)$ are finite. Now $0 \leq Y - X$ and applying Prop. III.1.4 we obtain $0 \leq E(Y - X) = E(Y) - E(X)$ which is the result.

(iii) Assume first that $X \geq 0$ and choose nonnegative simple $X_n \uparrow X$ Since $0 \leq X_n \leq X$ we have $P(X_n = 0) = 1$ and so by Prop. III.1.1(iii) $E(X_n) = 0$ which implies $E(X) = 0$ since $E(X_n) \to E(X)$. Now when $X = X_+ - X_-$, then $\{\omega : X(\omega) = 0\} \subset \{\omega : X_+(\omega) = 0\}$ (since $X_+(\omega) = X_-(\omega)$ only when both equal 0) and so $P(X_+ = 0) = 1$ which implies $E(X_+) = 0$ and similarly $E(X_-) = 0$ which implies $E(X) = 0$.

(iv) Suppose $P(X = Y) = 1$, put

$$A = \{\omega : X(\omega) = Y(\omega)\}, B = \{\omega : X(\omega) \neq Y(\omega)\}$$

so $P(B) = 0$. This implies $P(XI_B = 0) \geq P(B^c) = 1$ and so by (iii) $E(XI_B) = 0$. Also, $X = XI_A + XI_B$ and $Y = YI_A + YI_B = XI_A + YI_B$.

If $E(X)$ is finite, we have $E(XI_A) = E(X - XI_B) = E(X) - E(XI_B)$ by Prop. III.1.4 and so $E(X) = E(XI_A) = E(YI_A)$. By the same argument $E(YI_B) = 0$ and so by Prop. III.1.4

$$E(YI_A) = E(YI_A) + E(YI_B) = E(YI_A + YI_B) = E(Y)$$

and conclude that $E(X) = E(Y)$.

If $E(X)=\infty$, then $E(X_+)=\infty$ (and $E(X_-)$ is finite) which implies $E(X_+I_A)=\infty$ as $E(X_+I_B)=0$. Now $E(X_+I_A)=E(Y_+I_A)$ which implies $E(Y_+)=\infty$, otherwise $E(Y_+)<\infty$ would imply $E(Y_+I_A)=E(Y_+-Y_+I_B)=E(Y_+)-E(Y_+I_B)<\infty$. Since $E(X_-)$ is finite this implies $E(X_-)=E(Y_-)$ by the preceding argument and so $E(X)=E(Y)$. If $E(X)=-\infty$ apply the same argument to $-X$ and $-Y$. ■