Probability and Stochastic Processes I - Lecture 11

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II.7 Mutual Statistical Independence of Random Variables

- suppose we have a s.p. $\{(\lambda, X_\lambda) : \lambda \in \Lambda\}$

- what does it mean to say that the X_λ random variables are mutually statistically independent?

- recall

Definition 1.6.2 *When* $(\Omega, \mathcal{A}, P)$ *is a probability model and* $\{\mathcal{A}_\lambda : \lambda \in \Lambda\}$ *is a collection of sub* σ*-algebras of* $\mathcal{A}$*, then the* $\mathcal{A}_\lambda$ *are mutually statistically independent whenever* $\{\lambda_1, \ldots, \lambda_n\} \subset \Lambda$ *and for any* $A_1 \in \mathcal{A}_{\lambda_1}, \ldots, A_n \in \mathcal{A}_{\lambda_n}$*, then* $P(A_1 \cap \cdots \cap A_n) = \prod_{i=1}^n P(A_i)$.■

- also, for random variable X, then

$$\mathcal{A}_X = X^{-1}\mathcal{B}^1 = \{X^{-1}B : B \in \mathcal{B}^1\}$$

is a sub σ-algebra of $\mathcal{A}$ called the *σ-algebra generated by X*

Exercise II.7.1 Prove that $\mathcal{A}_X$ is a sub σ-algebra of $\mathcal{A}$.

Definition II.7.1 For the collection of random variables $\{X_\lambda : \lambda \in \Lambda\}$, the X_λ are mutually statistically independent if in the collection of σ-algebras $\{\mathcal{A}_{X_\lambda} : \lambda \in \Lambda\}$ the $\mathcal{A}_{X_\lambda}$ are mutually statistically independent. ■

Proposition II.7.1 For the collection of random variables $\{X_\lambda : \lambda \in \Lambda\}$, the X_λ are mutually statistically independent iff whenever $\{\lambda_1, \ldots, \lambda_n\} \subset \Lambda$, then the joint cdf of $(X_{\lambda_1}, \ldots, X_{\lambda_n})$ factors as the product of the marginal cdfs, namely, for every $(x_1, \ldots, x_n)$

$$F_{(X_{\lambda_1},\ldots,X_{\lambda_n})}(x_1, \ldots, x_n) = \prod_{i=1}^{n} F_{X_{\lambda_i}}(x_i).$$

Proof: $\Longrightarrow$) We have

$$\begin{aligned}
& F_{(X_{\lambda_1},\ldots,X_{\lambda_n})}(x_1, \ldots, x_n) \\
= \;& P_{(X_{\lambda_1},\ldots,X_{\lambda_n})}((-\infty, x_1] \times \cdots \times (-\infty, x_n]) \\
= \;& P(\{X_{\lambda_1} \in (-\infty, x_1]\} \cap \cdots \cap \{X_{\lambda_n} \in (-\infty, x_n]\}) \\
= \;& \prod_{i=1}^{n} P(\{X_{\lambda_i} \in (-\infty, x_i]\} = \prod_{i=1}^{n} F_{X_{\lambda_i}}(x_i).
\end{aligned}$$

$\Longleftarrow$) Recall that by the Extension Theorem the cdf $F_{(X_{\lambda_1},\ldots,X_{\lambda_n})}$ determines $P_{(X_{\lambda_1},\ldots,X_{\lambda_n})}$. Also $\prod_{i=1}^n F_{X_{\lambda_i}}$ arises as the cdf of the joint probability measure obtained by the product of the marginal probability measures $P_{X_{\lambda_i}}$ and so $X_{\lambda_1},\ldots,X_{\lambda_n}$ are mutually statistically independent. Clearly the collection of cdf's

$$\{\prod_{i=1}^n F_{X_{\lambda_i}} : \{\lambda_1,\ldots,\lambda_n\} \subset \Lambda \text{ for some } n\}$$

is consistent. By KCT this determines P_X and so the collection of random variables $\{X_\lambda : \lambda \in \Lambda\}$ are mutually statistically independent. ■

Proposition II.7.2 For the collection of random variables $\{X_\lambda : \lambda \in \Lambda\}$ and each $\{\lambda_1, \ldots, \lambda_n\} \subset \Lambda :$
(i) if each $(X_{\lambda_1}, \ldots, X_{\lambda_n})$ has a discrete distribution, then the X_λ are mutually statistically independent iff, for every $(x_1, \ldots, x_n)$,

$$p_{(X_{\lambda_1},\ldots,X_{\lambda_n})}(x_1, \ldots, x_n) = \prod_{i=1}^{n} p_{X_{\lambda_i}}(x_i),$$

(ii) if each $(X_{\lambda_1}, \ldots, X_{\lambda_n})$ has an a.c. distribution, then the X_λ are mutually statistically independent iff, for every $(x_1, \ldots, x_n)$,

$$f_{(X_{\lambda_1},\ldots,X_{\lambda_n})}(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{X_{\lambda_i}}(x_i).$$

Proof: **Exercise II.7.2.** ■

Example II.7.1 *Bernoulli(p) process*

- for any T and $\{t_1, \ldots, t_n\} \subset T$ then, for $(x_1, \ldots, x_n) \in \{0, 1\}^n$,

$$\begin{aligned} & p_{(X_{t_1}, \ldots, X_{t_n})}(x_1, \ldots, x_n) \\ = \; & p^{\sum_{i=1}^n x_i}(1-p)^{n-\sum_{i=1}^n x_i} \\ = \; & \prod_{i=1}^n p^{x_i}(1-p)^{1-x_i} \\ = \; & \prod_{i=1}^n p_{X_{t_i}}(x_i), \end{aligned}$$

with $X_{t_i} \sim$Bernoulli(p) and so by Prop. II.7.2 the X_λ are mut. stat. ind.
■

Example II.7.2 *Gaussian white noise process*

- for any T and $\{t_1, \ldots, t_n\} \subset T$ then, since

$$(X_{t_1}, \ldots, X_{t_n}) \sim N_n(\mathbf{0}, \text{diag}(\sigma^2(t_1), \ldots, \sigma^2(t_n)))$$

and for $(x_1, \ldots, x_n) \in R^n$,

$$\begin{aligned} & f_{(X_{\lambda_1},\ldots,X_{\lambda_n})}(x_1, \ldots, x_n) \\ = \; & (2\pi)^{-n/2}(\sigma^2(t_1)\cdots\sigma^2(t_n))^{-1/2} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{x_i^2}{\sigma^2(t_i)}\right) \\ = \; & \prod_{i=1}^{n}(2\pi)^{-1/2}\sigma^{-1}(t_i) \exp\left(-\frac{1}{2}\frac{x_i^2}{\sigma^2(t_i)}\right) \\ = \; & \prod_{i=1}^{n} f_{X_{t_i}}(x_i) \end{aligned}$$

with $X_{t_i} \sim N(0, \sigma^2(t_i))$ and so by Prop. II.7.2 the X_λ are mut. stat. ind. ■

Example II.7.3 *Principal components*

- suppose $\mathbf{X} \sim N_k(\boldsymbol{\mu}, \Sigma)$ where $\Sigma = Q\Lambda Q'$ (spectral decomposition)
- then $\mathbf{Y} = Q'\mathbf{X} \sim N_k(Q'\boldsymbol{\mu}, Q'\Sigma Q) = N_k(Q'\boldsymbol{\mu}, \Lambda)$ so

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{y}) &= \prod_{i=1}^{n} (2\pi)^{-1/2} \lambda_i^{-1/2} \exp\left(-\frac{1}{2}\frac{(y_i - \mathbf{q}_i'\boldsymbol{\mu})^2}{\lambda_i}\right) \\ &= \prod_{i=1}^{n} f_{Y_i}(y_i) \end{aligned}$$

with $Y_i = \mathbf{q}_i'\mathbf{X} = \sum_{j=1}^{k} q_{ji} X_j \sim N(\mathbf{q}_i'\boldsymbol{\mu}, \lambda_i) = N(\sum_{j=1}^{k} q_{ji}\mu_j, \lambda_i)$ and so the principal components $Y_1, \ldots, Y_k$ are mut. stat. ind. ■