Solutions to Exercises - Lecture 7

IV.4.1 (a) $\int_0^1\int_0^1\int_0^1 x_1 x_2 x_3 \, dx_1 \, dx_2 \, dx_3$

$= \int_0^1 x_1 \, dx_1 \int_0^1 x_2 \, dx_2 \int_0^1 x_3 \, dx_3 = \left(\frac{x_1^2}{2}\Big|_0^1\right)\left(\frac{x_2^2}{2}\Big|_0^1\right)\left(\frac{x_3^2}{2}\Big|_0^1\right)$

$= \frac{1}{2^3}$. Therefore $c = 8$ so $8x_1x_2x_3$ is a pdf on $[0,1]^3$.

(b) $P([\tfrac{1}{2},\tfrac{3}{4}]\times[\tfrac{2}{3},1]\times[0,\tfrac{1}{2}])$

$= 8\left(\frac{x_1^2}{2}\Big|_{1/2}^{3/4}\right)\left(\frac{x_2^2}{2}\Big|_{2/3}^{1}\right)\left(\frac{x_3^2}{2}\Big|_0^{1/2}\right)$

$= \left(\frac{9}{16}-\frac{1}{4}\right)\left(1-\frac{4}{9}\right)\left(\frac{1}{4}\right) = \left(\frac{5}{16}\right)\left(\frac{5}{9}\right)\left(\frac{1}{4}\right) = \frac{25}{576}$

(c) $\int_0^1\int_0^{x_3}\int_0^{x_2} x_1 x_2 x_3 \, dx_1 \, dx_2 \, dx_3$

$= \int_0^1\int_0^{x_3} \frac{x_1^2}{2}\Big|_0^{x_2} x_2 x_3 \, dx_2 \, dx_3$

$= \int_0^1\int_0^{x_3} \frac{x_2^3}{2} \, dx_2 \, x_3 \, dx_3 = \int_0^1 \frac{x_2^4}{8}\Big|_0^{x_3} x_3 \, dx_3$

$= \int_0^1 \frac{x_3^5}{8} \, dx_3 = \frac{x_3^6}{48}\Big|_0^1 = \frac{1}{48}$

Therefore $c = 48$.

$P([\tfrac{1}{2},\tfrac{3}{4}]\times[\tfrac{2}{3},1]\times[0,\tfrac{1}{2}])$

$= \int_0^{1/2}\int_{\frac{2}{3}}^{\min(x_3,1)}\int_{\frac{1}{2}}^{\min(x_2,3/4)} 48 x_1 x_2 x_3 \, dx_1 \, dx_2 \, dx_3$

$= 0$ since pdf $= 0$ when $x_3 < \frac{1}{2}$ and $x_2 > \frac{2}{3}$

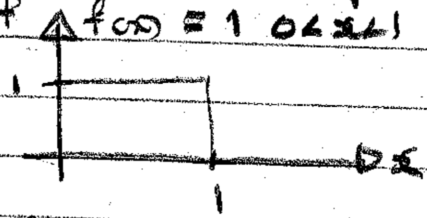
2.4.20 Clearly $f(x) \geq 0 \ \forall x$ and
$$\int_0^\infty \alpha(1+x)^{-(\alpha+1)}\,dx = -(1+x)^{-\alpha}\Big|_0^\infty = 0-(-1) = 1$$
and so f is a pdf.

2.4.21 Clearly $f(x) \geq 0 \ \forall x$ and
$$\int_{-\infty}^\infty \frac{1}{\pi}\frac{1}{1+x^2}\,dx = \frac{1}{\pi}\arctan x\Big|_{-\infty}^\infty = \frac{1}{\pi}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = 1$$
and so f is a pdf.

2.4.24 and 2.4.25 (a) Clearly $f(x) \geq 0$ and
$$\int_0^1 f(x)\,dx = \int_0^1 \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}\,dx = \frac{B(a,b)}{B(a,b)} = 1$$
and so f is a pdf.

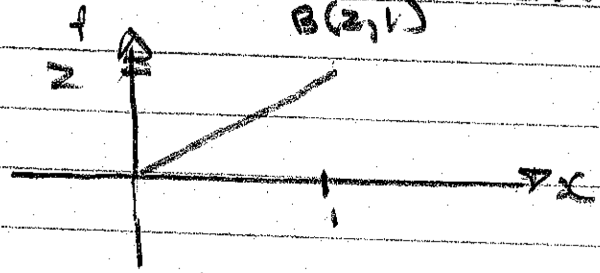
$f(x) = 1 \quad 0<x<1$

(b)

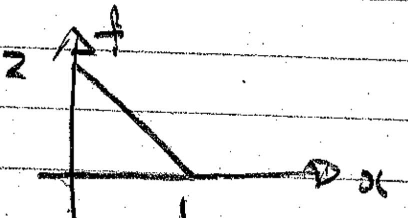


f is the uniform pdf

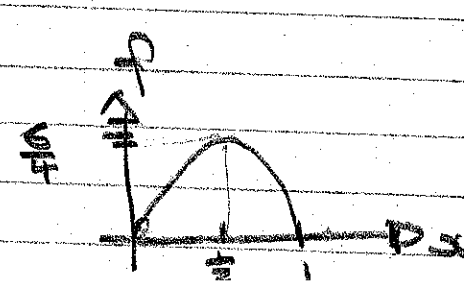
(c) $f(x) = \frac{x}{B(2,1)} = 2x \quad 0<x<1$



(d) $f(x) = 2(1-x) \quad 0<x<1$



(e) $f(x) = 6x(1-x)$



(2.4.25) Make the change of variable $\binom{u}{v} = T(x,y) = \begin{pmatrix} x+y \\ \frac{x}{x+y} \end{pmatrix}$

so $v = \frac{x}{u}$ which implies $x = uv$, $y = u - x = u(1-v)$ so

$T^{-1}(u,v) = \begin{pmatrix} uv \\ u(1-v) \end{pmatrix}$, $J_{T^{-1}}(u,v) = |\det\begin{pmatrix} v & u \\ 1-v & -u \end{pmatrix}|^{-1}$

$= |-uv - u(1-v)|^{-1} = u^{-1}$. Now $T\,[0,\infty) \times [0,\infty)$

$= [0,\infty) \times [0,1]$ and so $\Gamma(a)\,\Gamma(b)$

$= \int_0^\infty \int_0^\infty x^{a-1} y^{b-1} e^{-x-y}\, dx\, dy$

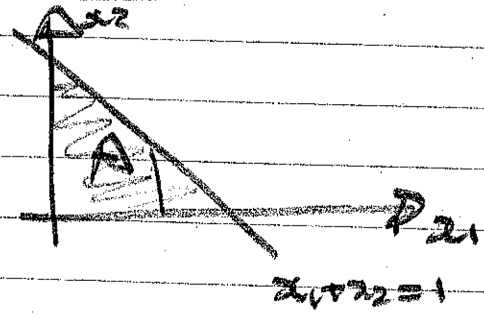
$= \int_0^\infty \int_0^1 (uv)^{a-1} u^{b-1} (1-v)^{b-1}\, u\, e^{-u}\, dv\, du$

$= \int_0^\infty u^{a+b-1} e^{-u}\, du \int_0^1 v^{a-1} (1-v)^{b-1}\, dv$

$= \Gamma(a+b)\, B(a,b)$ and the result is obtained.

2.7.17 and 2.7.18

(a)



Clearly $f(x_1,x_2) \geq 0$ and

$$\int_A f(x_1,x_2)\,dx_1\,dx_2 = \int_0^1 \int_0^{1-x_1} f(x_1,x_2)\,dx_2\,dx_1$$

$$= \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_0^1 x_1^{\alpha_1-1} \int_0^{1-x_1} x_2^{\alpha_2-1}(1-x_1-x_2)^{\alpha_3-1}\,dx_2\,dx_1$$

$$= \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_0^1 x_1^{\alpha_1-1}(1-x_1)^{\alpha_2+\alpha_3-2} \times$$

$$\left(\int_0^{1-x_1}\left(\frac{x_2}{1-x_1}\right)^{\alpha_2-1}\left(1-\frac{x_2}{1-x_1}\right)^{\alpha_3-1}dx_2\right)dx_1$$

make the change of variables
$x_2 \to y = \frac{x_2}{1-x_1}$ so $dy = \frac{dx_2}{1-x_1}$

$$= \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_0^1 x_1^{\alpha_1-1}(1-x_1)^{\alpha_2+\alpha_3-1}\,dx_1$$

$$\int_0^1 y^{\alpha_2-1}(1-y)^{\alpha_3-1}\,dy$$

by previous Exercise

$$= \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \frac{\Gamma(\alpha_1)\Gamma(\alpha_2+\alpha_3)}{\Gamma(\alpha_1+\alpha_2+\alpha_3)} \frac{\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_2+\alpha_3)} = 1$$

and f is a pdf.

(b) $f_{X_1}(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\,dx_2$

since $0 < x_1 + x_2 < 1$ $= \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} x_1^{\alpha_1-1} \int_0^{1-x_1} x_2^{\alpha_2-1}(1-x_1-x_2)^{\alpha_3-1}\,dx_2$

and make the change of variable as in (a) part

$= \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} x_1^{\alpha_1-1}(1-x_1)^{\alpha_2+\alpha_3-1} \int_0^1 y^{\alpha_2-1}(1-y)^{\alpha_3-1}\,dy$

$= \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \frac{\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_2+\alpha_3)} x_1^{\alpha_1-1}(1-x_1)^{\alpha_2+\alpha_3-1}$

$= \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2+\alpha_3)} x_1^{\alpha_1-1}(1-x_1)^{\alpha_2+\alpha_3-1}$ for $0 < x_1 < 1$

Therefore $X_1 \sim \text{beta}(\alpha_1, \alpha_2+\alpha_3)$ and similarly $X_2 \sim \text{beta}(\alpha_2, \alpha_1+\alpha_3)$.

2.7.18 Clearly $f_{(X_1,\ldots,X_k)}(x_1,\ldots,x_k) \geq 0$ and

$\int_0^1 \int_0^{1-x_k} \cdots \int_0^{1-x_3-\cdots-x_k} \int_0^{1-x_2-\cdots-x_k} f(x_1,\ldots,x_k)\,dx_1\,dx_2 \cdots dx_k$

$= \frac{\Gamma(\alpha_1+\cdots+\alpha_{k+1})}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_{k+1})} \int_0^1 \int_0^{1-x_k} \cdots \int_0^{1-x_3-\cdots-x_k} x_2^{\alpha_2-1}\cdots x_k^{\alpha_k-1} \times$

$\left(\int_0^{1-x_2-\cdots-x_k} x_1^{\alpha_1-1}(1-x_1-x_2-\cdots-x_k)^{\alpha_{k+1}-1}\,dx_1\right) dx_2 \cdots dx_k$

and making the change of variable $x_1 \to y = \frac{x_1}{1-x_2-\cdots-x_k}$

$$= \frac{\Gamma(\alpha_1+\cdots+\alpha_{k+1})}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_{k+1})} \int_0^1 \int_0^{1-x_k} \cdots \int_0^{1-x_3-\cdots-x_k} x_2^{\alpha_2-1}\cdots x_k^{\alpha_k-1}(1-x_2-\cdots-x_k)^{\alpha_1+\alpha_{k+1}-1} \times$$

$$\underbrace{\left(\int_0^1 y^{\alpha_1-1}(1-y)^{\alpha_{k+1}-1}\,dy\right)}_{\shortparallel \atop \frac{\Gamma(\alpha_1)\Gamma(\alpha_{k+1})}{\Gamma(\alpha_1+\alpha_{k+1})}} dx_2\cdots dx_k$$

$$= \frac{\Gamma(\alpha_1+\cdots+\alpha_{k+1})}{\Gamma(\alpha_1+\alpha_{k+1})\Gamma(\alpha_2)\cdots\Gamma(\alpha_k)} \int_0^1 \int_1^{1-x_k} \cdots x_3^{\alpha_3-1}\cdots x_k^{\alpha_k-1} \times$$

$$\left(\int_0^{1-x_3-\cdots-x_k} x_2^{\alpha_2-1}(1-x_2-\cdots-x_k)^{\alpha_1+\alpha_{k+1}-1}\,dx_2\right)$$

$$dx_3\cdots dx_k$$

making the change of variable

$$x_2 \to \frac{y}{1-x_3-\cdots-x_k}$$

$$= \frac{\Gamma(\alpha_1+\cdots+\alpha_{k+1})}{\Gamma(\alpha_1+\alpha_{k+1})\Gamma(\alpha_2)\cdots\Gamma(\alpha_k)} \int_0^1 \int_0^{1-x_k} \cdots x_3^{\alpha_3-1}\cdots x_k^{\alpha_k-1}(1-x_3-\cdots-x_k)^{\alpha_1+\alpha_2+\alpha_{k+1}-1} \times$$

$$\frac{\Gamma(\alpha_2)\Gamma(\alpha_1+\alpha_{k+1})}{\Gamma(\alpha_1+\alpha_2+\alpha_{k+1})}\, dx_3\cdots dx_k$$

$\vdots$ as above

$$= \frac{\Gamma(\alpha_1+\cdots+\alpha_{k+1})}{\Gamma(\alpha_1+\cdots+\alpha_{k-1}+\alpha_{k+1})\Gamma(\alpha_k)} \int_0^1 x_k^{\alpha_k-1}(1-x_k)^{\alpha_1+\cdots+\alpha_{k-1}+\alpha_{k+1}-1}\,dx_k$$

$= 1$ and so f is a pdf.

II.4.6 Consider (X_1, X_2). Then

$$P_{(X_1,X_2)}(x_1,x_2) = P_{(X_1,\ldots,X_k)}(\{x_1\}\times\{x_2\}\times R\times\cdots\times R)$$

$$= \sum_{x_3\in R,\ldots,x_k\in R} P_{(X_1,\ldots,X_k)}(x_1,x_2,x_3,\ldots,x_k)$$

Similarly for any subvector $(X_{i_1},\ldots,X_{i_l})$ the prob. fn of $P_{(X_{i_1},\ldots,X_{i_l})}$ at $(x_{i_1},\ldots,x_{i_l})$ is obtained by summing the joint prob fn $P_{(X_1,\ldots,X_k)}$ over all possible values of the remaining variables.

II.4.7 $P_{X_1}(x_1) = \binom{n}{x_1} p_1^{x_1}(1-p_1)^{n-x_1}$, $x_1\in\{0,\ldots,n\}$

as $X_1 \sim$ binomial (n, p_1) and

$$P_{(X_1,X_2)}(x_1,x_2) = \binom{n}{x_1\ x_2\ n-x_1-x_2} p_1^{x_1} p_2^{x_2}(1-p_1-p_2)^{n-x_1-x_2}$$

for $x_1, x_2 \in \{0,\ldots,n\}$ with $x_1+x_2 \le n$.

In general marginals for the multinomial are multinomial with the same n and p_i but with a final category containing the remaining counts and occurring with the remaining prob.

II.4.8 Following the calculations in the first step of 2.7.18 of Ex II.4.5 we see that, after integrating out x_1 the result
$(x_2, \ldots, x_n) \sim \text{Dirichlet}(\alpha_2, \ldots, \alpha_n, \alpha_1 + \alpha_{n+1})$
is obtained. Clearly it doesn't matter in which order we integrate out variables so if we had integrated out x_n first we would obtain the result

$$(x_1, x_2, \ldots, x_{n-1}) \sim \text{Dirichlet}(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n + \alpha_{n+1})$$

Then iterating this procedure we obtain

$$(X_1, X_2) \sim \text{Dirichlet}(\alpha_1, \alpha_2, \alpha_3 + \cdots + \alpha_{n+1})$$

$$X_1 \sim \text{Dirichlet}(\alpha_1, \alpha_2 + \cdots + \alpha_{n+1})$$

$$= \text{beta}(\alpha_1, \alpha_2 + \cdots + \alpha_{n+1}).$$

The general result is that after integrating out any subset of the variables the marginal distribution of the remaining variables is also Dirichlet with the same α_i excepting the last which is the sum of α_{n+1} plus the α's associated with the variables that were integrated out.