Solutions to Exercises - Lecture 5

VI.1.1 Let $\omega \in X^{-1}(B_1 \cup B_2)$. Then $X(\omega) \in B_1 \cup B_2$ so $X(\omega) \in B_1$ or $X(\omega) \in B_2$ which implies $\omega \in X^{-1}B_1$ or $\omega \in X^{-1}B_2$ which implies $\omega \in X^{-1}B_1 \cup X^{-1}B_2$. Therefore $X^{-1}(B_1 \cup B_2) \subseteq X^{-1}B_1 \cup X^{-1}B_2$. Now suppose $\omega \in X^{-1}B_1 \cup X^{-1}B_2$ so $\omega \in X^{-1}B_1$ or $\omega \in X^{-1}B_2$ so $X(\omega) \in B_1$ or $X(\omega) \in B_2$ which implies $X(\omega) \in B_1 \cup B_2$ which implies $\omega \in X^{-1}(B_1 \cup B_2)$ which implies $X^{-1}B_1 \cup X^{-1}B_2 \subseteq X^{-1}(B_1 \cup B_2)$. Therefore $X^{-1}(B_1 \cup B_2) = X^{-1}B_1 \cup X^{-1}B_2$.

Let $\omega \in X^{-1}B^c$ then $X(\omega) \in B^c$ which implies $X(\omega) \notin B$ which implies $\omega \notin X^{-1}B$ which implies $\omega \in (X^{-1}B)^c$ so $X^{-1}B^c \subseteq (X^{-1}B)^c$. Now suppose $\omega \in (X^{-1}B)^c$ so $\omega \notin X^{-1}B$ which implies $X(\omega) \notin B$ which implies $X(\omega) \in B^c$ which implies $\omega \in X^{-1}B^c$ and so $(X^{-1}B)^c \subseteq X^{-1}B^c$. Therefore $X^{-1}B^c = (X^{-1}B)^c$

The proof of $X^{-1}(A \cap B) = X^{-1}A \cap X^{-1}B$ is similar. Now suppose $A \cap B = \phi$. Then $X^{-1}A \cap X^{-1}B = X^{-1}(A \cap B) = X^{-1}\phi = \phi$.

VI.1.2 Suppose $n \in \mathbb{Z}$ is even. Then

$$X^{-1}(-\infty, b] = \begin{cases} \phi & , b < 0 \\ [-b^{1/n}, b^{1/n}] & , b > 0 \end{cases} \in \mathcal{B}^1.$$

If $n \in \mathbb{Z}$ is odd then

$$X^{-1}(-\infty, b] = \begin{cases} (-\infty, -|b|^{1/n}] & \text{if } b < 0 \\ (-\infty, b^{1/n}] & \text{if } b > 0 \end{cases}$$

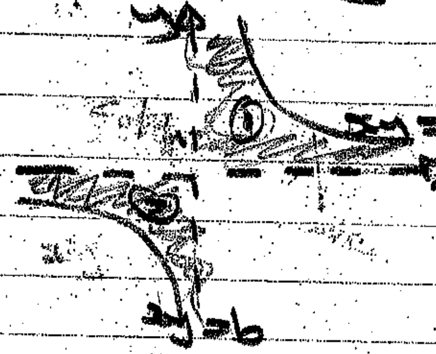
$\in \mathcal{B}^1$. Therefore by Prop. VI.1.2 X is a r.v.

(II.1.3) Suppose $b=0$. Then $W^{-1}(-\infty,0]$

$= \{\omega : X(\omega)\le 0,\ Y(\omega)\ge 0\} \cup \{\omega : X(\omega)\ge 0,\ Y(\omega)\le 0\}$

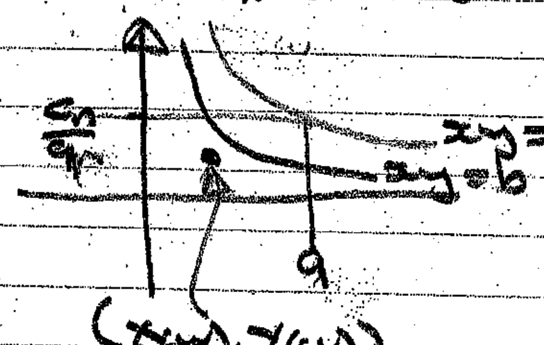
$= (X^{-1}(-\infty,0]\cap Y^{-1}[0,\infty)) \cup (X^{-1}[0,\infty)\cap Y^{-1}(-\infty,0]) \in \mathcal{A}$.

Now suppose $b>0$. Then $W^{-1}(-\infty,b]$
$= W^{-1}(-\infty,0] \cup W^{-1}(0,b]$ and since $W^{-1}(-\infty,0]\in\mathcal{A}$
we need only show that $W^{-1}(0,b]\in\mathcal{A}$



$= \{\omega : X(\omega)>0,\ Y(\omega)>0,\ X(\omega)Y(\omega)\le b\}$
$\cup\ \{\omega : X(\omega)<0,\ Y(\omega)<0,\ X(\omega)Y(\omega)\le b\}$
$=$ ① $\cup$ ②. Clearly if we show ① $\in\mathcal{A}$ then the same argument will show ② $\in\mathcal{A}$ and we are done this case

Let $c_n \downarrow b$ and suppose $\omega \in$ ①. Then $\exists\, q\in\mathbb{Q}\cap(0,\infty)$



s.t. $\omega\in X^{-1}(0,q]\cap Y^{-1}(0,\frac{c_n}{q}]$ $\in\mathcal{A}$ and putting

$$C_n = \bigcup_{q\in\mathbb{Q}\cap(0,\infty)} X^{-1}(0,q]\cap Y^{-1}(0,\tfrac{c_n}{q}]$$

$\in\mathcal{A}$ since $\mathbb{Q}\cap(0,\infty)$ is countable. Clearly $C_n \downarrow$ ① and thus ① $\in\mathcal{A}$ and this completes the argument for $b>0$. The argument for $b<0$ is basically the same and so we have proved that $W=XY$ is a r.v.

IV.1.4 By the class notes any constant fn $Y(\omega) = c$ is a r.v. Then by Prop. IV.1.3 (ii) $a_i X_1^i$ is a r.v. for each $i = 0, 1, 2, \ldots, n$ since products of r.v.'s are r.v.'s. Then by Prop IV.1.3 (i) $p(x)$ is a r.v. since sums of r.v.'s are r.v.'s.

IV.1.5 This follows exactly as in IV.1.4.

IV.1.6 $\underset{\sim}{X}^{-1}\{(0,1)\} = \{1,2\}$ $\underset{\sim}{X}^{-1}\{(1,0)\} = \{3\}$
(note typo: last entry should have been $X_2(3) = 0$)

Thus for $B \in \mathcal{B}^2$ we have $\underset{\sim}{X}^{-1}B = \begin{cases} \phi & \text{if } (0,1),(1,0) \notin B \\ \{1,2\} & \text{if } (0,1) \in B, (1,0) \notin B \\ \{3\} & \text{if } (1,0) \in B, (0,1) \notin B \\ \underline{\Omega} & \text{otherwise} \end{cases}$

and so

$$P_{\underset{\sim}{X}}(B) = \begin{cases} 0 & \text{if } (0,1), (1,0) \notin B \\ 5/6 & \text{if } (0,1) \in B, (1,0) \notin B \\ 1/6 & \text{if } (0,1) \notin B, (1,0) \in B \\ 1 & \text{otherwise.} \end{cases}$$

IV.1.7 (as part of Prop IV.1.4)

Let $\mathcal{B}_* = \{ B \times \underbrace{\mathbb{R}^1 \times \cdots \times \mathbb{R}^1}_{k-1 \text{ times}} : B \in \mathcal{B}^1 \}$. Since $(-\infty, b] \times \mathbb{R}^1 \times \cdots \times \mathbb{R}^1 \in \mathcal{B}^k$ $\forall b \in \mathbb{R}^1$ this implies $B \times \mathbb{R}^1 \times \cdots \times \mathbb{R}^1 \in \mathcal{B}^k$ $\forall B \in \mathcal{B}^1$. Now (i) $\phi \times \mathbb{R}^1 \times \cdots \times \mathbb{R}^1 = \phi \in \mathcal{B}^*$, (ii) if $B_i \times \mathbb{R}^1 \times \cdots \times \mathbb{R}^1 \in \mathcal{B}_*$ for $i = 1, 2, \ldots$ then $\bigcup_{i=1}^{\infty} B_i \times \mathbb{R}^1 \times \cdots \times \mathbb{R}^1 = (\bigcup_{i=1}^{\infty} B_i) \times \mathbb{R}^1 \times \cdots \times \mathbb{R}^1 \in \mathcal{B}_*$ since $\bigcup_{i=1}^{\infty} B_i \in \mathcal{B}^1$ (iii) if $B \times \mathbb{R}^1 \times \cdots \times \mathbb{R}^1 \in \mathcal{B}_*$

then $(B \times \mathbb{R}^1 \times \cdots \times \mathbb{R}^1)^c = B^c \times \mathbb{R}^1 \times \cdots \times \mathbb{R}^1 \in \mathcal{B}_*$ since $B^c \in \mathcal{B}^1$. This proves that $\mathcal{B}_*$ is a σ-algebra