Solutions - Lecture 4

(1.7.1) Since $\phi = C \cap \phi$ for any event C and $0 = P(\phi) = P(\phi)P(C)$ we have that ϕ and C are always stat. ind. Similarly $C = C \cap \Omega$ so $P(C) = P(C \cap \Omega) = P(C)P(\Omega)$ since $P(\Omega) = 1$ and so Ω is stat. ind. of C.

Now $A \cap B^c = A \cap (A \cap B)^c = A \setminus (A \cap B)$ and so $P(A \cap B^c) = P(A) - P(A \cap B)$ since $A \cap B \subseteq A$ $= P(A) - P(A)P(B)$ since A and B are stat. ind. $= P(A)(1 - P(B)) = P(A)P(B^c)$ since $P(B^c) = 1 - P(B)$ and so A and B^c are stat. ind. Similarly A^c and B are stat. ind, A and B^c are stat. ind. Finally $P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B)$ $= 1 - P(A) - P(B) + P(A \cap B) = 1 - P(A) - P(B) - P(A)P(B)$ $= (1 - P(A))(1 - P(B)) = P(A^c)P(B^c)$ and A^c and B^c are stat. ind.

(1.7.2) (a) Consider $\mathcal{A}_1 = \{\phi, \{1\} \times \{1,2\}, \{2\} \times \{1,2\}, \{1,2\}^2\}$. We need to check that $\mathcal{A}_1$ is closed under complementation and unions. Clearly $\phi^c = \{1,2\} \times \{1,2\}$, $(\{1,2\} \times \{1,2\})^c = \phi$, $(\{1\} \times \{1,2\})^c = \{(x_1, x_2) : x_1 \neq 1\} = \{(2,1), (2,2)\}$ $= \{2\} \times \{1,2\} \in \mathcal{A}_1$. Similarly $(\{2\} \times \{1,2\})^c = \{1\} \times \{1,2\} \in \mathcal{A}_1$. Also $(\{1\} \times \{1,2\}) \cup (\{2\} \times \{1,2\}) = \{(1,1), (1,2)\} \cup \{(2,1), (2,2)\}$ $= \{(1,1), (1,2), (2,1), (2,2)\} = \{1,2\} \times \{1,2\} \in \mathcal{A}_1$.
All the other unions are trivially in $\mathcal{A}_1$ and so we have proved that $\mathcal{A}_1$ is a σ-algebra. Similarly $\mathcal{A}_2$ is a σ-algebra.
(b) $P(\{1\} \times \{1,2\}) = P(\{1,2\} \times \{2\}) = \frac{1}{2}$ and $P((\{1\} \times \{1,2\}) \cap (\{1,2\} \times \{2\})) = P(\{(1,2)\})$ $= \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$ which implies stat. ind for all $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$ by Exercise 1.6.1.

1.8.1, E&R 1.3.8 A = "female", B = "has long hair"
$P(A) = 0.55$ $P(A \cap B) = 0.44$ and $P(A^c \cap B) = 0.15$.
So $B = (A \cap B) \cup (A^c \cap B)$ and $(A \cap B) \cap (A^c \cap B) = \phi$
and thus $P(B) = P(A \cap B) + P(A^c \cap B) = 0.44 + 0.15$
$= 0.59$. Therefore $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 0.55 + 0.59 - 0.44 = 0.70$.

1.8.2 $P(A \cup B \cup C) = P((A \cup B) \cup C)$
$= P(A \cup B) + P(C) - P((A \cup B) \cap C)$
$= P(A) + P(B) - P(A \cap B) + P(C) - P((A \cap C) \cup (B \cap C))$
$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C)$
$+ P((A \cap C) \cap (B \cap C))$
and since $(A \cap C) \cap (B \cap C) = A \cap B \cap C$
$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C)$
$+ P(A \cap B \cap C)$

1.8.3 $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) +$
$\sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(A_1 \cap \cdots \cap A_n)$

Proof: (by induction). The result is true for $n=2$. Now assume true for n and consider

$P(A_1 \cup \cdots \cup A_n \cup A_{n+1}) = P((A_1 \cup \cdots \cup A_n) \cup A_{n+1})$
$\overset{n=2}{=} P(A_1 \cup \cdots \cup A_n) + P(A_{n+1}) - P((A_1 \cup \cdots \cup A_n) \cap A_{n+1})$
$\overset{(1)}{=} P(A_1 \cup \cdots \cup A_n) + P(A_{n+1}) - P((A_1 \cap A_{n+1}) \cup \cdots \cup (A_n \cap A_{n+1}))$

Now apply inductive step to obtain

$P(A_1 \cup \cdots \cup A_n)$

$\overset{②}{=} \sum_{i=1}^{n} P(A_i) - \sum_{i<j\le n} P(A_i \wedge A_j) + \cdots + (-1)^{n+1} P(A_1 \wedge \cdots \wedge A_n)$

$P((A_1 \wedge A_{n+1}) \cup \cdots \cup (A_n \wedge A_{n+1}))$

$\overset{③}{=} \sum_{i=1}^{n} P(A_i \wedge A_{n+1}) - \sum_{i<j\le n} P(A_i \wedge A_j \wedge A_{n+1}) + \cdots +$

$(-1)^{n+1} P(A_1 \wedge \cdots \wedge A_n \wedge A_{n+1})$

Combining ①, ② and ③ gives

$P(A_1 \cup \cdots \cup A_{n+1})$

$= \sum_{i=1}^{n+1} P(A_i) - \sum_{i<j} P(A_i \wedge A_j) + \cdots + (-1)^{n+2} P(A_1 \wedge \cdots \wedge A_{n+1})$

which proves the result.

1.84 $P(A \wedge B \wedge C) = 1 - P((A \wedge B \wedge C)^c)$

$= 1 - P(A^c \cup B^c \cup C^c)$

$= 1 - [P(A^c) + P(B^c) + P(C^c) - P(A^c \wedge B^c) - P(A^c \wedge C^c)$

$- P(B^c \wedge C^c) + P(A^c \wedge B^c \wedge C^c)]$

$= 1 - [3 - P(A) - P(B) - P(C) - (1 - P(A \cup B))$

$- (1 - P(A \cup B)) - (1 - P(B \cup C)) + (1 - P(A \cup B \cup C))]$

$= P(A)+P(B)+P(C) - P(A\cup B) - P(A\cup C) - P(B\cup C)$
$+ P(A\cup B\cup C)$

So in general

$$P(A_1 \wedge \cdots \wedge A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cup A_j) + \cdots + (-1)^{n+1} P(A_1 \cup \cdots \cup A_n)$$

The formulas in Exercises 1.7.3 and 1.7.4 are known as the Inclusion - Exclusion formulas.

(1.8.5) Let $x \in \liminf_{n\to\infty} (-\frac{1}{n}, 1+\frac{n-1}{n}]$
$= \bigcup_{n=1}^{\infty} \bigcap_{r=n}^{\infty} (-\frac{1}{r}, 1+\frac{r-1}{r}]$ then $\exists\, n_0$ s.t.
$x \in \bigcap_{r=n_0}^{\infty} (-\frac{1}{r}, 1+\frac{r-1}{r}]$ which implies $-\frac{1}{r} < x \le 1+\frac{r-1}{r} < 2$
$\forall\, r \ge n_0$ which implies $0 \le x < 2$.

Also if $x \in [0,2)$ then $\exists\, n_0$ s.t. $-\frac{1}{i} < x \le 1+\frac{i-1}{i}$
for every $i \ge n_0$ and so $x \in \liminf_{n\to\infty} (-\frac{1}{n}, 1+\frac{n-1}{n}]$

Therefore $\liminf_{n\to\infty} (-\frac{1}{n}, 1+\frac{n-1}{n}] = [0,2)$

Similarly $\limsup_{n\to\infty} (-\frac{1}{n}, 1+\frac{n-1}{n}] = [0,2)$ and

therefore $\lim_{n\to\infty} (-\frac{1}{n}, 1+\frac{n-1}{n}] = [0,2)$.

So the limiting prob. is (by the cty of P)

$\lim P\left(\left(-\frac{1}{n}, 1-\frac{n-1}{n}\right]\right) = \Phi(2) - \Phi(0)$

where Φ is the cdf of the $N(0,1)$.

1.8.6 EJR 1.6.10

(a) $A_n = (0, \frac{1}{n})$ is a monotone decreasing sequence of sets and so $\lim A_n = \bigcap_{n=1}^{\infty} (0, \frac{1}{n})$ and this is the null set since for $x \le 0$ $x \notin (0, \frac{1}{n})$ for any n and for $x > 0$ $\exists n_0$ s.t. $1/n < x$ $\forall n \ge n_0$ so $x \notin (0, 1/n)$.
Therefore by continuity of P
$P(\lim A_n) = P(\phi) = 0$.

(b) Let P be the prob. measure on (R^1, B^1) given by $P(B) = 1$ when $0 \in B$ and $P(B) = 0$ otherwise. This is called the prob. measure degenerate at 0.
Clearly $P([0, 1/n)) = 1$ $\forall n$ and $\lim_{n\to\infty} [0, 1/n) = \{0\}$. So $P(\{0\}) = P(\lim [0, 1/n))$

$= \lim P([0, 1/n)) = 1$.

Ex 1.8.7, E+R 1.6.11

Let $A_1, A_2, \ldots \in \mathcal{A}$ be mut. disjoint and consider $B_n = \bigcup_{i=1}^{n} A_i$. Then B_n is a monotone increasing seq. of events with $\lim B_n = \bigcup_{n=1}^{\infty} B_n = \bigcup_{i=1}^{\infty} A_i$.

Also

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P(\lim B_n) \overset{\text{cty of } P}{=} \lim P(B_n)$$

$$= \lim P\left(\bigcup_{i=1}^{n} A_i\right) \overset{\text{finite addty}}{=} \lim \sum_{i=1}^{n} P(A_i)$$

$$= \sum_{i=1}^{\infty} P(A_i)$$

which proves countable additivity.