Exercises Lecture 17 - Solutions

Ex VI.6.1 $\mathrm{cov}(a_0 + \sum_{i=1}^{m} a_i X_i ,\ b_0 + \sum_{j=1}^{n} b_j Y_j)$

$= \mathrm{cov}(a_0 + \underset{\sim}{a}'\underset{\sim}{X},\ b_0 + \underset{\sim}{b}'\underset{\sim}{Y})$

$= \underset{\sim}{a}' \mathrm{cov}(\underset{\sim}{X}, \underset{\sim}{Y})\, \underset{\sim}{b}$ by Ex VI.5.2 (ii)

where $\underset{\sim}{a} = (a_1, \ldots, a_m)'$,
$\underset{\sim}{b} = (b_1, \ldots, b_n)'$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j \mathrm{cov}(X_i, Y_j)$$

Ex VI.6.2 $\mathrm{Var}(a_0 + \sum_{i=1}^{n} a_i X_i)$

$= \mathrm{Var}(a_0 + \underset{\sim}{a}\underset{\sim}{X}) = \underset{\sim}{a}' \mathrm{Var}(\underset{\sim}{X})\, \underset{\sim}{a}$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \mathrm{cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} a_i^2 \mathrm{Var}(X_i) + 2 \sum_{i<j} \mathrm{cov}(X_i, X_j)$$

and when the $X_1, \ldots, X_n$ are mut. stat. ind

then $\mathrm{cov}(X_i, X_j) = 0$ when $i \neq j$ so

$$\mathrm{Var}(a_0 + \sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i \mathrm{Var}(X_i)$$

Ex III, 6.3

In the case of the Gaussian random walk

$$\underset{\sim}{X} \sim N_t\left(A\, m \underset{\sim}{1}_t,\ A\tau^2 I A'\right)$$

where $\underset{\sim}{1}_t = (\overbrace{1, \ldots, 1}^{t \text{ times}})'$ and

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & & \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \end{pmatrix} \quad \text{so}$$

$$\underset{\sim}{X} \sim N_t\left(\begin{pmatrix} m \\ 2m \\ \vdots \\ tm \end{pmatrix},\ \tau^2 \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2 \\ \vdots & & & \\ 1 & 2 & \cdots & t \end{pmatrix}\right)$$

In the case of Example III, 6.4

$$\underset{\sim}{X} \sim N_{n+1}\left(A\underset{\sim}{0},\ A\tau^2 A'\right) \text{ where}$$

$$A = \begin{pmatrix} \theta & 1 & 0 & \cdots & 0 \\ 0 & \theta & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & & \theta & 1 \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+2)}$$

Therefore $\underset{\sim}{X} \sim N_{n+1}\left(\underset{\sim}{0},\ \tau^2 \begin{pmatrix} 1+\theta^2 & \theta & 0 & \cdots & 0 \\ \theta & 1+\theta^2 & \theta & 0 & \cdots 0 \\ \vdots & \theta & \ddots & \ddots & \\ 0 & \cdots & & \theta & 1+\theta^2 \end{pmatrix}\right)$