Exercise Lecture 16 - Solutions Part II

Ex III.5.1 $\mathrm{cov}(h_1(\underset{\sim}{x}), h_2(\underset{\sim}{y}))$

$= E(h_1(\underset{\sim}{x})\, h_2(\underset{\sim}{y})) - E(h_1(\underset{\sim}{x}))E(h_2(\underset{\sim}{y})) = 0$

since $E(h_1(\underset{\sim}{x})h_2(\underset{\sim}{y})) = E(h_1(\underset{\sim}{x}))E(h_2(\underset{\sim}{y}))$

by the independence of $h_1(\underset{\sim}{x})$ and $h_2(\underset{\sim}{y})$

and Prop. III.5.1

Ex III.5.2

(i) $\mathrm{cov}(\underset{\sim}{x}, \underset{\sim}{y}) = (\mathrm{cov}(x_i, y_j))$ ((i,j)-th entry of matrix)

and $\mathrm{cov}(x_i, y_j) \in R$ provided $E(x_i y_j) \in R^1$

$E(x_i) \in R^1$ and $E(y) \in R^1$ which are

guaranteed to be real when $E(x_i^2) < \infty$,

$E(y_j^2) < \infty \;\; \forall i, j.$

(ii) $\mathrm{cov}(\underset{\sim}{a} + A\underset{\sim}{x}, \underset{\sim}{b} + B\underset{\sim}{y})$

$= E((\underset{\sim}{a} + A\underset{\sim}{x} - E(\underset{\sim}{a} + A\underset{\sim}{x}))(\underset{\sim}{b} + B\underset{\sim}{y} - E(\underset{\sim}{b} + B\underset{\sim}{y}))')$

$= E((\underset{\sim}{a} + A\underset{\sim}{x} - \underset{\sim}{a} - AE(\underset{\sim}{x}))(\underset{\sim}{b} + B\underset{\sim}{y} - \underset{\sim}{b} - BE(\underset{\sim}{y}))')$

$= A\, E((\underset{\sim}{x} - E(\underset{\sim}{x}))(\underset{\sim}{y} - E(\underset{\sim}{y}))')\, B' = A\, \mathrm{cov}(\underset{\sim}{x}, \underset{\sim}{y})\, B'$

(iii) $Cov(\underline{X}, \underline{Y}) = (Cov(X_i, Y_j))$ and since $\underline{X}, \underline{Y}$ are independent then by Prop III.5.1 $h_1(\underline{X}), h_2(\underline{Y})$ are independent for any $h_1 : (R^k, B^k) \to (R^1, B^1)$, $h_2 : (R^l, B^l) \to (R^1, B^1)$. Letting $h_1(\underline{X}) = X_i$, $h_2(\underline{Y}) = Y_j$ we have that X_i, Y_j are independent and so $Cov(X_i, Y_j) = 0$. Therefore $Cov(\underline{X}, \underline{Y}) = 0 \in R^{k \times l}$ (the $k \times l$ matrix of 0's).

Ex VII.5.3 (i) $\text{CORR}(\underset{\sim}{X}) = D_{\underset{\sim}{X}}^{-1} \Sigma_X D_{\underset{\sim}{X}}^{-1}$

$$= \begin{pmatrix} 1/SD(x_1) & & 0 \\ & \ddots & \\ 0 & & 1/SD(x_k) \end{pmatrix} \begin{pmatrix} COV(x_1,x_1) & COV(x_1,x_2) & \cdots & COV(x_1,x_k) \\ COV(x_2,x_1) & COV(x_2,x_2) & & \vdots \\ \vdots & & \ddots & \\ COV(x_k,x_1) & \cdots & \cdots & COV(x_k,x_k) \end{pmatrix} \times \begin{pmatrix} 1/SD(x_1) & & 0 \\ & \ddots & \\ 0 & & 1/SD(x_k) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{COV(x_1,x_1)}{SD(x_1)} & \frac{COV(x_1,x_2)}{SD(x_1)} & \cdots & \frac{COV(x_1,x_k)}{SD(x_1)} \\ \frac{COV(x_2,x_1)}{SD(x_2)} & & & \\ \vdots & & & \\ \frac{COV(x_k,x_1)}{SD(x_k)} & \cdots & \cdots & \frac{COV(x_k,x_k)}{SD(x_k)} \end{pmatrix} \begin{pmatrix} 1/SD(x_1) & & 0 \\ & \ddots & \\ 0 & & 1/SD(x_k) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{COV(x_1,x_1)}{SD(x_1)SD(x_1)} & \frac{COV(x_1,x_2)}{SD(x_1)SD(x_2)} & \cdots & \frac{COV(x_1,x_k)}{SD(x_1)SD(x_k)} \\ \vdots & & & \\ \frac{COV(x_k,x_1)}{SD(x_k)SD(x_1)} & \cdots & \cdots & \frac{COV(x_k,x_k)}{SD(x_k)SD(x_k)} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & CORR(x_1,x_2) & \cdots & CORR(x_1,x_k) \\ \vdots & 1 & & \\ \vdots & & \ddots & \\ CORR(x_1,x_k) & \cdots & & 1 \end{pmatrix}$$

Since $COV(x_i,x_i)/SD(x_i)SD(x_i) = Var(x_i)/Var(x_i)$

$= 1$ and $COV(x_i,x_j)/SD(x_i)SD(x_j) = CORR(x_i,x_j)$.

(ii) $\operatorname{Var}(D\underset{\sim}{X}) = D\operatorname{Var}(\underset{\sim}{X})D' = D\operatorname{Var}(\underset{\sim}{X})D$

and so $\operatorname{Var}(Y_i) = d_i^2 \operatorname{Var}(X_i)$ which implies

$SD(Y_i) = d_i\, SD(X_i)$. Therefore

$$\operatorname{CORR}(\underset{\sim}{Y}) = \begin{pmatrix} d_1 SD(X_1) & & \\ & \ddots & \\ & & d_k SD(X_k) \end{pmatrix}^{-1} \operatorname{Var}(D\underset{\sim}{X}) \left(\qquad\right)^{-1}$$

$$= \left(D\begin{pmatrix} SD(X_1) & & 0 \\ & \ddots & \\ 0 & & SD(X_k) \end{pmatrix}\right)^{-1} D\operatorname{Var}(\underset{\sim}{X})D \left(\qquad\right)^{-1}$$

$$= \begin{pmatrix} SD(X_1) & & \\ & \ddots & \\ & & SD(X_k) \end{pmatrix}^{-1} D^{-1} D \operatorname{Var}(\underset{\sim}{X}) D D^{-1} \left(\qquad\right)^{-1}$$

$$= \begin{pmatrix} SD(X_1) & & 0 \\ & \ddots & \\ 0 & & SD(X_k) \end{pmatrix}^{-1} \operatorname{Var}(\underset{\sim}{X}) \left(\qquad\right)^{-1}$$

$$= \operatorname{CORR}(\underset{\sim}{X})$$

(iii) Suppose $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

then $\operatorname{CORR}(\underset{\sim}{X}) = \begin{pmatrix} 1 & \sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}} \\ \sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}} & 1 \end{pmatrix}$ and

$$\operatorname{Var}(\underset{\sim}{Y}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ -\sigma_{12} & -\sigma_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_{11} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{22} \end{pmatrix} \text{ so } \operatorname{CORR}(\underset{\sim}{Y}) = \begin{pmatrix} 1 & -\sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}} \\ -\sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}} & 1 \end{pmatrix} \neq \operatorname{CORR}(\underset{\sim}{X}).$$